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Matrix Representation of Kauffman Networks

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Abstract: Kauffman networks are a class of Boolean networks where each node has the same number of incoming connections. Despite the simplicity of such networks, they exhibit very complex behaviors and have been shown to be an appropriate model for certain gene regulatory networks. Kauffman networks are typically represented by Boolean logics for which no efficient analytical tools are available. The logical representation of Kauffman networks makes it extremely difficult to analyze their dynamic behaviors. Based on a recently developed tool named "semi-tensor product" for matrices, we propose a novel matrix representation for Kauffman networks. This matrix representation is essentially a linear discrete dynamic system, making it possible to analyze the dynamic behaviors of Kauffman networks using existing tools in dynamic systems. As an example of the advantages of using this matrix representation, we show how the number and length of attractors can be calculated efficiently which is an impossible task for the original logical representation. Some general properties of Kauffman networks are also discussed based on their matrix representation.

Key Words: Semi-tensor product, Kauffman networks, Matrix representation

1 Introduction

Kauffman networks, also referred to as N-K networks or random Boolean networks, were first introduced by Kauffman in 1969 [1] as a simplified model for gene regulatory networks. Since its first introduction, persistent efforts have been devoted to it; see recent works in [2–6] and the references therein. Although the main interests in this model are due to gene regulatory networks, the understanding of other complex networks such as the small world network also benefits from this simple yet meaningful model [7].

Kauffman networks are constructed by randomly choosing the Boolean logics and the same number of incoming connections for each node. One is usually interested in the mean properties of the ensemble of all the possible Kauffman networks with certain system size n and incoming connections k. This ensemble-based analysis can offer us the knowledge of how the system structure, i.e., the system size n and the incoming connections k, would affect the network properties of interest in the mean sense, for example, the mean number and length of attractors [8–10]. However, Kauffman networks are typically represented by a number of interacted logical functions for which no analytical tools are available. To make it worse, the possible states of Kauffman networks are increasing with the power of 2^n with the system size n and the ensemble of Kauffman networks is increasing even faster, making it impossible to analyze such networks using simulation-based approaches even for as few as dozens of nodes. Therefore, novel approaches to dealing with Kauffman networks have been sought for decades.

To deal with this difficulty, we propose a novel matrix representation for Kauffman networks. This is obtained following recent works on the matrix representation of general Boolean networks based on a novel tool called "semi-tensor product" [11, 12], a generalized product of matrices. Using semi-tensor product, Boolean networks can be represented by a linear discrete dynamic system. This allows us to analyze the dynamic behaviors of Boolean networks using tools in discrete dynamic system theory. We consider specific restrictions of Kauffman networks compared to general Boolean networks and reflect these restrictions in the matrix representation of Kauffman networks. We also discuss some further properties of Kauffman networks based on their matrix representation and as an example of the advantages of the matrix representation we show how to calculate the number and length of attractors of a specific Kauffman network based on its matrix representation.

The remainder of the paper is organized as follows. For completeness we first introduce the matrix representation of general Boolean networks in Section 2. We then apply the general matrix representation to Kauffman networks in Section 3. This involves both determining whether a given matrix representation of a Boolean network is a Kauffman network or not and generating the matrix representation of Kauffman networks from their logical representation. More properties and the efficient way to calculate the number and length of attractors of Kauffman networks are discussed based on the matrix representation in Section 4 and Section 5 concludes the paper.

2 The matrix representation of general Boolean networks

We first give the definition of semi-tensor product for completeness.

Definition 1 (Semi-tensor product, [11]) *For any matrices X and Y with dimensions $r_1 \times c_1$ and $r_2 \times c_2$, the semi-tensor product of X and Y, denoted by $X \ltimes Y$, is defined as follows,*

$$X \ltimes Y := (X \otimes I_{\mathrm{lcm}(c_1,r_2)/c_1})(Y \otimes I_{\mathrm{lcm}(c_1,r_2)/r_2}) \quad (1)$$

where $\mathrm{lcm}(c_1, r_2)$ is the least common multiple of c_1 and r_2 and $\otimes$ represents the Kronecker product.

It is noticed that semi-tensor product is a generalization of normal product of matrices and therefore in what follows

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we might omit the symbol $\ltimes$ wherever no confusion can be caused.

Let $\mathcal{D} := \{1, 0\}$, where $1 \sim T$ and $0 \sim F$ represent "True" and "False", respectively. We can use vectors to represent the logical values in $\mathcal{D}$, as follows,

$$T \sim 1 \sim \delta_2^1, \ F \sim 0 \sim \delta_2^2 \tag{2}$$

where δ_n^k denotes the kth column of the identity matrix with dimension n, I_n. Let

$$\Delta_n := \{\delta_n^k | 1 \leq k \leq n\} \tag{3}$$

For simplicity of notations, let $\Delta_2 := \Delta$ and then $\Delta \sim \mathcal{D}$. An $n \times m$ matrix M is called a logical matrix if

$$M = [\delta_n^{i_1} \ \delta_n^{i_2} \ \ldots \ \delta_n^{i_m}] \tag{4}$$

which can be denoted simply by

$$M = \delta_n[i_1 \ i_2 \ \ldots \ i_m] \tag{5}$$

where $1 \leq i_p \leq n, p = 1, 2, \ldots, m$.

Denote the set of all $n \times m$ logical matrices by $\mathcal{L}_{n \times m}$.

We have the following fundamental result based on semi-tensor product [12].

Theorem 1 ([12]) *Let $f(x_1, x_2, \ldots, x_n)$ be a logical function. There exists a unique $M \in \mathcal{L}_{2 \times 2^n}$, called the structure matrix of f, such that*

$$f(x_1, x_2, \ldots, x_n) = M \ltimes_{i=1}^n x_i \tag{6}$$

Now consider a Boolean network with n nodes, as follows,

$$\begin{cases} x_1(t+1) = f_1(x_1(t), \ldots, x_n(t)) \\ \vdots \\ x_n(t+1) = f_n(x_1(t), \ldots, x_n(t)) \end{cases} \tag{7}$$

According to Theorem 1, the Boolean network in (7) can be equivalently represented in its component-wise matrix representation, as follows,

$$\begin{cases} x_1(t+1) = L_1 x(t) \\ \vdots \\ x_n(t+1) = L_n x(t) \end{cases} \tag{8}$$

Let $x(t) := \ltimes_{i=1}^n x_i(t)$. The above component-wise matrix representation can be further rewritten in a compact form, as follows,

$$x(t+1) = Lx(t) \tag{9}$$

with

$$L = L_1 * L_2 * \ldots * L_n \tag{10}$$

where $*$ is the Khatri-Rao product. That is,

$$\mathrm{Col}_i(L) = \ltimes_{j=1}^n \mathrm{Col}_i(L_j), i = 1, \ldots, 2^n \tag{11}$$

Remark 1 *The logical functions we considered here are in the functionally equivalent sense. That is, two logical functions are the same if and only if they are functionally equivalent. For example, although the two logical functions, $f(x_1, x_2) = x_1$ and $g(x_1, x_2) = (x_1 \wedge x_2) \vee (x_1 \wedge \neg x_2)$ where $\wedge$, $\vee$ and $\neg$ represent conjunction, disjunction and negation, respectively, are different in their expressions, they are the same in terms of their functionalities as the same input can guarantee the same output. They are regarded as one function. In this sense, the mapping of the logical functions from their logical representation to their matrix representation is bijective and thus we are free to use the matrix representation in all cases.*

3 The matrix representation of Kauffman networks

We give a formal definition of classic Kauffman networks, as follows.

Definition 2 (Classic Kauffman networks) *A classic Kauffman network, $NK(n, k)$, is a Boolean network with n nodes in which each node receives a fixed number of k incoming connections.*

In classic Kauffman networks the requirement of the same number of incoming connections for each node is a tight restriction. We may want to consider a generalized version by relaxing this restriction, as follows.

Definition 3 (Generalized Kauffman networks) *A generalized Kauffman network, $GNK(n, k)$, is a Boolean network with n nodes in which each node receives no more than k incoming connections.*

In both the above definitions, the incoming connections and corresponding logical functions are chosen randomly for each node when the network is constructed but they remain fixed during the dynamics of the network. As all the results that follows for classic Kauffman networks can be readily extended to generalized Kauffman networks, we may thus either omit the results for generalized Kauffman networks or simply present the results without proof for brevity.

3.1 Determining whether a Boolean network in its matrix representation is a Kauffman network

Definition 4 *Given a logical function f. A node is said to be free to f if the output of f is entirely independent of the state of this node. The number of nodes that are free to f is denoted by $\kappa(f)$.*

Based on Definition 4, we are able to give an alternative definition for Kauffman networks (classic and generalized).

Proposition 1 (Alternative definition of Kauffman networks) *A Boolean network with size n is a classic Kauffman network, $NK(n, k)$, if and only if*

$$\kappa(f_j) = n - k, \forall 1 \leq j \leq n \tag{12}$$

It is a generalized Kauffman network, $GNK(n, k)$, if and only if

$$\kappa(f_j) \geq n - k, \forall 1 \leq j \leq n \tag{13}$$

Proof. Straightforward by definitions.

Construct $S_i^n \in \mathcal{L}_{2\times 2^n}$ by 2^i blocks with equal size of $2 \times 2^{n-i}$ and the odd and even blocks being $\delta_2[1, 1, \ldots, 1]$ and $\delta_2[2, 2, \ldots, 2]$, respectively, and define

$$P_i^n := \{j | \operatorname{Col}_j(S_i^n) = \delta_2^1\}, \bar{P}_i^n := \{j | \operatorname{Col}_j(S_i^n) = \delta_2^2\} \tag{14}$$

Note that we also keep the ascending order of the elements in P_i^n and $\bar{P}_i^n$. For example, $P_1^3 = \{1, 2, 3, 4\}$, $\bar{P}_1^3 = \{5, 6, 7, 8\}$.

We have the following criterion to determine whether a node is free to a logical function.

Proposition 2 *Given the structure matrix M of a logical function f on x. Node x_i is free to f if and only if*

$$M_i = \bar{M}_i \tag{15}$$

where

$$M_i = M \ltimes \delta_{2^i}[P_i^i],\ \bar{M}_i = M \ltimes \delta_{2^i}[\bar{P}_i^i] \tag{16}$$

where $\delta_{2^i}[P_i^i]$ and $\delta_{2^i}[\bar{P}_i^i]$ are the logical matrices with their jth columns being $\delta_{2^i}^{P_i^i(j)}$ and $\delta_{2^i}^{\bar{P}_i^i(j)}$, respectively.

Proof. x_i is free to f if and only if the following two matrices are identical: one is constructed by the columns of M belonging to P_i^n and the other by the columns of M belonging to $\bar{P}_i^n$. These two matrices are exactly those defined above by the definition of semi-tensor product.

The following theorem offers a way to determine whether a Boolean network given its component-wise matrix representation is a classic Kauffman network. Note the similar criteria applies to generalized Kauffman networks as well. We omit the discussions for brevity.

Theorem 2 *A Boolean network with size n is a classic Kauffman network, $NK(n, k)$, if and only if for each of its logical functions f_j, there exist and only exist k indexes, $1 \le i_1, i_2, \ldots, i_k \le n$, such that*

$$(L_j)_{i_l} = (\bar{L}_j)_{i_l}, l = 1, 2, \ldots, k \tag{17}$$

where

$$(L_j)_{i_l} = L_j \ltimes \delta_{2^{i_l}}[P_{i_l}^{i_l}],\ (\bar{L}_j)_{i_l} = L_j \ltimes \delta_{2^{i_l}}[\bar{P}_{i_l}^{i_l}] \tag{18}$$

Proof. Straightforward by Propositions 1 and 2.

As the component-wise matrix representation in (8) of a Boolean network can be readily obtained from its compact matrix representation in (9), Proposition 2 and Theorem 2 thus apply to Boolean networks in the compact matrix representation as well.

Corollary 1 *Given a Boolean network in* (9). *Node x_i is free to function f_j if and only if either of the following two cases is true*

$$1.\mathbb{L}(P_i^n) \subseteq P_j^n, \mathbb{L}(\bar{P}_i^n) \subseteq \bar{P}_j^n; \tag{19}$$

$$2.\mathbb{L}(P_i^n) \subseteq \bar{P}_j^n, \mathbb{L}(\bar{P}_i^n) \subseteq P_j^n \tag{20}$$

where $\mathbb{L}(P_i^n)$ and $\mathbb{L}(\bar{P}_i^n)$ denote the values in L in the columns belonging to P_i^n and $\bar{P}_i^n$, respectively.

Proof. Notice that such columns in L_j with the values being 1 (or 2) will make the values in L belong to P_j^n (or $\bar{P}_j^n$). The result is thus straightforward by Proposition 2.

The criterion of determining whether a given Boolean network is a Kauffman network in the compact matrix representation is given as follows.

Theorem 3 *Given a Boolean network in* (9). *It is a classic Kauffman network, $NK(n, k)$, if and only if*

$$\arg\{i | \textit{either } (19) \textit{ or } (20) \textit{ is true}\} = k, \forall 1 \le j \le n \tag{21}$$

Proof. Straightforward from Theorem 2 and Proposition 1.

Example 1 *Suppose we have only the following component-wise matrix representation of a Boolean network,*

$$\begin{cases} x_1(t+1) = \delta_2[1\ 1\ 2\ 2\ 1\ 1\ 2\ 2]x(t) \\ x_2(t+1) = \delta_2[1\ 1\ 1\ 1\ 2\ 2\ 2\ 2]x(t) \\ x_3(t+1) = \delta_2[2\ 2\ 1\ 1\ 2\ 2\ 1\ 1]x(t) \end{cases}$$

It is easy to check that $(L_1)_1 = (\bar{L}_1)_1$, $(L_1)_3 = (\bar{L}_1)_3$ and $(L_1)_2 \ne (\bar{L}_1)_2$. Therefore $\kappa(f_1) = 2$. Similarly we have $\kappa(f_2) = \kappa(f_3) = 2$ and therefore it is a classic Kauffman network with $NK(3, 1)$ by Theorem 2.

If we know only the compact matrix representation

$$x(t+1) = \delta_8[2\ 2\ 5\ 5\ 4\ 4\ 7\ 7]x(t)$$

we can check that there exist and only exist the following inclusion relationships,

$$\mathbb{L}(P_2^3) \subseteq P_1^3, \mathbb{L}(P_1^3) \subseteq P_2^3, \mathbb{L}(P_2^3) \subseteq P_3^3$$

It is thus a classic Kauffman network with $NK(3, 1)$ by Theorem 3.

3.2 Kauffman networks: From logical representation to matrix representation

The logical representation of classic Kauffman networks, $NK(n, k)$ can be written as follows,

$$\begin{cases} x_1(t+1) = f_1(x_{i_{11}}(t), \ldots, x_{i_{1k}}(t)) \\ \vdots \\ x_n(t+1) = f_n(x_{i_{n1}}(t), \ldots, x_{i_{nk}}(t)) \end{cases} \tag{22}$$

where $i_{j1}, i_{j2}, \ldots, i_{jk}$ are the indexes of the nodes that function f_j depends on. Without loss of generality we assume these indexes are in the ascending order. For simplicity of notations we let $[i]_j^{n,k} := \{i_{j1}, i_{j2}, \ldots, i_{jk}\}$ denote the sequence in the ascending order constructed by choosing k elements from $\{1, 2, \ldots, n\}$. As n, k have been reserved to denote the two parameters in Kauffman networks, in what follows we may simply use $[i]_j$ (or even $[i]$ when function f_j is not explicitly focused on) wherever no confusion can be caused.

Let $x_{[i]_j} := \ltimes_{l=1}^k x_{i_{jl}}$ and denote by $L_{[i]_j}$ the structure matrix for f_j on $x_{[i]_j}$. The component-wise matrix representation of (22) can be written as

$$\begin{cases} x_1(t+1) = L_{[i]_1} x_{[i]_1}(t) \\ \vdots \\ x_n(t+1) = L_{[i]_n} x_{[i]_n}(t) \end{cases} \tag{23}$$

Theorem 4 shows how we can construct the structure matrix for f_j on x, denoted by L_j, from the available structure matrix on $x_{[i]_j}$, $L_{[i]_j}$.

Theorem 4 *Suppose the structure matrix of a logical function f, M^k, depends on a subset of k nodes out of the n nodes, $\mathcal{V}_s^k := x_{[i]} = \{x_{i_1}, \ldots, x_{i_k}\}$. Its structure matrix dependent on all the nodes $\mathcal{V} := \{x_1, \ldots, x_n\}$, denoted by M, can be obtained as follows.*

$$M = M^k T_{[i]} \tag{24}$$

where

$$T_{[i]} := \ltimes_{l=1}^{n-k} T_{i_{k+l}} \tag{25}$$

$$T_{i_{k+l}} = \delta_{2^{i_{k+l}-1}}[1\ 1\ 2\ 2\ \ldots\ 2^{i_{k+l}-1}\ 2^{i_{k+l}-1}],\ x_{i_{k+l}} \in \mathcal{V}_m^{n-k} \tag{26}$$

where $\mathcal{V}_m^{n-k} := \mathcal{V} \backslash \mathcal{V}_s^k = \{x_{i_{k+1}}, \ldots, x_{i_n}\}$ is the set of all the missing nodes in f in the ascending order.

Proof. M can be constructed by inserting the missing nodes from $\mathcal{V}_m^{n-k}$ one by one into M^k.

Suppose the first $l-1$ nodes in $\mathcal{V}_m^{n-k}$ have been inserted into M^k and the new structure matrix dependent on these $k+l-1$ nodes is denoted by M^{k+l-1}. The sets of the missing and the already selected nodes are now $\mathcal{V}_m^{n-k-l+1} = \{x_{i_{k+l}}, \ldots, x_{i_n}\}$ and $\mathcal{V}_s^{k+l-1} = \{x_{i_1}, \ldots, x_{i_{k+l-1}}\}$, respectively. Now consider the lth node in $\mathcal{V}_m^{n-k}$, i_{k+l}, which is also the first in $\mathcal{V}_m^{n-k-l+1}$, and the i_{k+l}th in the selected nodes set $\mathcal{V}_s^{k+l} = \{x_{i_1}, \ldots, x_{i_{k+l}}\}$. As all the nodes with index less than i_{k+l} have already been inserted, M^{k+l} can thus be constructed as follows.

1) Split M^{k+l-1} as blocks with size $2^{k+l-i_{k+l}}$.
2) For all these $2^{i_{k+l}-1}$ blocks in M^{k+l-1}, copy each of such blocks right behind it.

It is readily seen from Proposition 2 that M^{k+l} depends only on $\mathcal{V}_s^{k+l}$. The above procedure corresponds to

$$M^{k+l} = M^{k+l-1} \ltimes T_{i_{k+l}}$$

Noticing that $M = M^n$ and the above procedure can be done repeatedly, (24) readily follows.

Denote $\mathcal{V}_m^{[i]_j}$ the set of missing variables in function f_j, and $T_{[i]_j} := \ltimes_{i_l \in \mathcal{V}_m^{[i]_j}} T_{i_l}$. Then

$$\begin{cases} x_1(t+1) = L_{[i]_1} T_{[i]_1} x(t) \\ \vdots \\ x_n(t+1) = L_{[i]_n} T_{[i]_n} x(t) \end{cases} \tag{27}$$

and

$$x(t+1) = Lx(t) \tag{28}$$

where

$$L = L_{[i]_1} T_{[i]_1} * L_{[i]_2} T_{[i]_2} * \ldots * L_{[i]_n} T_{[i]_n} \tag{29}$$

Remark 2 *For generalized Kauffman networks, $L_{[i]_j}$ can be arbitrary. However for classic Kauffman networks, $L_{[i]_j}$ may not be arbitrary due to the requirement that for each node there exists exactly k incoming connections. $L_{[i]_j}$ must establish itself to be dependent on all the k chosen nodes.*

Corollary 2 *$T_{[i]}$ is uniquely defined by n and $[i]$, and*

$$T_{[i]} \in \mathcal{L}_{2^{i_n - n + k} \times 2^{i_n}} \tag{30}$$

Furthermore, in classic Kauffman networks there are $\begin{pmatrix} n \\ k \end{pmatrix}$ possible choices of $T_{[i]}$ with equal probability.

Proof. Straightforward.

If $T_{[i]}$ is known, the structure matrix M can be obtained as follows.

Proposition 3 *Suppose*

$$T_{[i]} = \delta_{2^{i_n - n + k}}[j_1\ j_2\ \ldots\ j_{2^{i_n}}] \tag{31}$$

Then

$$M = [M^k \delta_{2^{i_n - n + k}}^{j_1}\ \ldots\ldots\ M^k \delta_{2^{i_n - n + k}}^{j_{2^{i_n}}}] \tag{32}$$

with

$$M^k \delta_{2^{i_n - n + k}}^{j_l} = [M^k((j_l - 1)2^{n - i_n} + 1)\ \ldots\ M^k(j_l 2^{n - i_n})]$$

Proof. Notice that $n - k \le i_n \le n$, thus

$$0 \le i_n - n + k \le k$$

Notice also $M^k \in \mathcal{L}_{2 \times 2^k}$ and $T_{[i]} \in \mathcal{L}_{2^{i_n - n + k} \times 2^{i_n}}$, and therefore

$$M^k T_{[i]} = M^k (T_{[i]} \otimes I_{2^{n - i_n}})$$

The result readily follows from definitions.

Remark 3 *From* (27) *it is readily seen that the two factors affecting the dynamics of Kauffman networks are clearly identified. $T_{[i]_j}$ represents the random choice of k incoming connections, with each choice having the same probability of $\begin{pmatrix} n \\ k \end{pmatrix}$. $L_{[i]_j}$ represents the logical function on these selected k incoming connections, with each possible option having the same probability of 2^{2^k}.*

Example 2 *Consider Example 1 again. Its logical representation is*

$$\begin{cases} x_1(t+1) = x_2(t) \\ x_2(t+1) = x_1(t) \\ x_3(t+1) = \neg x_2(t) \end{cases}$$

Its component-wise matrix representation on k nodes can be obtained as follows,

$$\begin{cases} x_1(t+1) = \delta_2[1\ 2] x_2(t) \\ x_2(t+1) = \delta_2[1\ 2] x_1(t) \\ x_3(t+1) = \delta_2[2\ 1] x_2(t) \end{cases}$$

Using Theorem 4, we are able to obtain the component-wise matrix representation on all the nodes, as follows,

$$\begin{cases} x_1(t+1) = \delta_2[1\ 1\ 2\ 2\ 1\ 1\ 2\ 2] x(t) \\ x_2(t+1) = \delta_2[1\ 1\ 1\ 1\ 2\ 2\ 2\ 2] x(t) \\ x_3(t+1) = \delta_2[2\ 2\ 1\ 1\ 2\ 2\ 1\ 1] x(t) \end{cases}$$

Furthermore, the compact matrix representation is

$$x(t+1) = \delta_8[2\ 2\ 5\ 5\ 4\ 4\ 7\ 7] x(t)$$

4 Properties of Kauffman networks and its application to attractor analysis

In this Section we discuss the properties of the matrix representation of Kauffman networks and as an example of the advantages of using this matrix representation we show how efficiently the number and length of attractors of Kauffman networks can be calculated.

4.1 Properties of Kauffman networks

We organize the properties of classic Kauffman networks in the following theorem. Similar results can be found for generalized Kauffman networks but are omitted for brevity.

Theorem 5 *Suppose f_j is one of the logical functions in a classic Kauffman network, $NK(n,k)$. Its structure matrix on all the nodes is denoted by L_j which actually depends only on k nodes with indexes $[i]_j = \{i_{j1}, \ldots, i_{jk}\}$. Then*

1) *The number of free columns in L_j is 2^k.*
2) *Define*

$$\mathcal{Q}_j^{n,k} := \{\cap_{l=1}^{k} (P_{i_{jl}}^n)^{\pm} | (P_{i_{jl}}^n)^{\pm} = P_{i_{jl}}^n \text{ or } \bar{P}_{i_{jl}}^n\} \quad (33)$$

Then

$$\mathrm{Card}(\mathcal{Q}_j^{n,k}) = 2^k \quad (34)$$

$$\cup_{Q_{ji} \in \mathcal{Q}_j^{n,k}} Q_{ji} = \{1, \ldots, 2^n\} \quad (35)$$

$$\mathrm{Card}(Q_{ji}) = 2^{n-k}, \forall Q_{ji} \in \mathcal{Q}_j^{n,k} \quad (36)$$

$$Q_{j1} \cup Q_{j2} = \emptyset, \forall Q_{j1}, Q_{j2} \in \mathcal{Q}_j^{n,k} \quad (37)$$

$$\mathbb{L}_j(Q_{ji}) = \delta_2[1 \ldots 1] \text{ or } \delta_2[2 \ldots 2] \; \forall Q_{ji} \in \mathcal{Q}_j^{n,k} \quad (38)$$

$$\mathbb{L}(Q_{ji}) \subseteq P_j^n \text{ or } \bar{P}_j^n, 1 \le i \le 2^k \quad (39)$$

where Card *represents the cardinality of a set and $\mathbb{L}_j(Q_{ji})$ and $\mathbb{L}(Q_{ji})$ are the values in L_j and L in the columns belonging to Q_{ji}, respectively.*

3) *If the number of nodes that serve as inputs (denoted by n_d) is less than n, then in L the columns can be divided into 2^{n_d} blocks with the size of each block being 2^{n-n_d}. In each of the blocks, the values are the same.*

Proof. From Theorem 4 it is seen that L_j is generated by repeating different sizes of blocks from $L_{[i]_j}$ and thus L_j and $L_{[i]_j}$ have the same number of free columns, i.e., 2^k. This proves the first part of the Theorem.

Since $\mathcal{Q}_j^{n,k}$ contains the intersections of k sets and each set has two options, and it is easy to check that any such two intersections are different, (34) is thus true. The correctness of the results in (35) through (39) can be checked similarly by definitions and noticing Proposition 1 and we omit the details of proof.

The third part of the Theorem is true by using(39).

Example 3 *Consider a Kauffman network, $NK(5,2)$, as follows,*

$$\begin{cases} x_1(t+1) = \delta_2[1\ 2\ 1\ 1] x_2(t) x_3(t) \\ x_2(t+1) = \delta_2[1\ 2\ 2\ 2] x_1(t) x_4(t) \\ x_3(t+1) = \delta_2[2\ 1\ 1\ 2] x_2(t) x_5(t) \\ x_4(t+1) = \delta_2[1\ 2\ 1\ 1] x_1(t) x_2(t) \\ x_5(t+1) = \delta_2[1\ 2\ 1\ 1] x_1(t) x_3(t) \end{cases}$$

Its component-wise matrix representation on all the nodes is obtained as (by Theorem 4)

$$\begin{cases} x_1(t+1) = \delta_2[11112222111111111111222211111111] x(t) \\ x_2(t+1) = \delta_2[11221122112211222222222222222222] x(t) \\ x_3(t+1) = \delta_2[21212121121212122121212112121212] x(t) \\ x_4(t+1) = \delta_2[11111111222222221111111111111111] x(t) \\ x_5(t+1) = \delta_2[11112222111122221111111111111111] x(t) \end{cases}$$

Then we are able to obtain its compact matrix representation, as follows,

$$x(t+1) = \delta_{32}[5\ 1\ 13\ 9\ 22\ 18\ 30\ 26\ 3\ 7\ 11\ 15\ 4\ 8\ 12\ 16 \\ 13\ 9\ 13\ 9\ 29\ 25\ 29\ 25\ 9\ 13\ 9\ 13\ 9\ 13\ 9\ 13\ ] x(t) \quad (40)$$

In what follows we show (39) *in Theorem 5 is correct (for $j = 1$). The correctness of other parts of the Theorem can be checked similarly.*

By definition we obtain

$$\begin{aligned} \mathcal{Q}_1^{5,2} = \{ & Q_{11} = P_2^5 \cap P_3^5 = \{1,2,3,4,17,18,19,20\}, \\ & Q_{12} = P_2^5 \cap \bar{P}_3^5 = \{5,6,7,8,21,22,23,24\}, \\ & Q_{13} = \bar{P}_2^5 \cap P_3^5 = \{9,10,11,12,25,26,27,28\}, \\ & Q_{14} = \bar{P}_2^5 \cap \bar{P}_3^5 = \{13,14,15,16,29,30,31,32\}\} \end{aligned}$$

Then (39) *is true since*

$$\mathbb{L}(Q_{11}) = \{5,1,13,9,13,9,13,9\} \subseteq P_1^5 = \{1,2,\ldots,16\}$$
$$\mathbb{L}(Q_{12}) = \{22,18,30,26,29,25,29,25\} \subseteq \bar{P}_1^5 = \{17,\ldots,32\}$$
$$\mathbb{L}(Q_{13}) = \{3,7,11,15,9,13,9,13\} \subseteq P_1^5 = \{1,2,\ldots,16\}$$
$$\mathbb{L}(Q_{14}) = \{4,8,12,16,9,13,9,13\} \subseteq P_1^5 = \{1,2,\ldots,16\}$$

4.2 Calculating the attractors for Kauffman networks using its matrix representation

The number and length of attractors is one of the most important properties in a Boolean network. It is extremely difficult, if not impossible, to obtain this information in the logical representation. However, the matrix representation offers us a complete solution to this problem [11].

Theorem 6 ([11]) *Given a Boolean network in the compact matrix representation with the structure matrix being L. The number of attractors of length s, denoted by N_s, is inductively determined by*

$$N_1 = \mathrm{trace}(L) \quad (41)$$

$$N_s = \frac{\mathrm{trace}(L^s) - \sum_{k \in \mathcal{P}(s)} k N_k}{s}, 2 \le s \le 2^n \quad (42)$$

where $\mathcal{P}(s)$ is the set of proper factors of s.

Example 4 *We show how Theorem 6 can be used for Kauffman networks in Example 3. In fact, if the compact matrix representation of the considered Kauffman network is known, then the calculation of the number of attractors of different lengths is a sdandard procedure.*

Based on L *in* (40) *and applying Theorem 6, it is easy to obtain*

$$N_1 = 2, N_2 = 1, N_3 = 0, N_4 = 1, N_i = 0, i \geq 5$$

This result can never be readily obtained based on the conventional logical representation.

5 Conclusions

Kauffman networks are one of the most important models in Boolean networks which is useful as a simplified model for various complex systems. In order to deal with the difficulty of the lack of analytical tools in the conventional logical representation of Kauffman networks, we propose a matrix representation motivated by recent works on the matrix representation of general Boolean networks based on the novel tool called semi-tensor product. This matrix representation of Kauffman networks is essentially a linear discrete dynamic system for which extensive tools are available. It is believed that this novel matrix representation will open up a new direction of dealing with Kauffman networks and fruitful results are expected in the near future.

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1. Matrix representation of Kauffman networks

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Abstract: Kauffman networks are a class of Boolean networks where each node has the same number of incoming connections. Despite the simplicity of such networks, they exhibit very complex behaviors and have been shown to be an appropriate model for certain gene regulatory networks. Kauffman networks are typically represented by Boolean logics for which no efficient analytical tools are available. The logical representation of Kauffman networks makes it extremely difficult to analyze their dynamic behaviors. Based on a recently developed tool named 'semi-tensor product' for matrices, we propose a novel matrix representation for Kauffman networks. This matrix representation is essentially a linear discrete dynamic system, making it possible to analyze the dynamic behaviors of Kauffman networks using existing tools in dynamic systems. As an example of the advantages of using this matrix representation, we show how the number and length of attractors can be calculated efficiently which is an impossible task for the original logical representation. Some general properties of Kauffman networks are also discussed based on their matrix representation. © 2013 TCCT, CAA.
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