

Robust model predictive control for constrained networked nonlinear systems: An approximation-based approach [☆]



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ABSTRACT

In this paper, a robust approximation-based model predictive control (RAMPC) scheme for the constrained networked control systems (NCSs) subject to external disturbances is proposed. At each sampling instant, the approximate discrete-time model (DTM) is utilized for solving the optimal control problem online, and the control input applied to continuous-time systems can then be determined. Such RAMPC scheme enables to implement MPC for the continuous-time systems in the digital environment, and meanwhile, achieves the state and control input constraints satisfaction in continuous-time sense. Furthermore, we also provide a guideline to determine the allowable sampling period. Sufficient conditions for the feasibility of the RAMPC scheme as well as the associated stability are developed. Finally, the effectiveness of the RAMPC scheme is shown through a numerical simulation.

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1. Introduction

Networked control systems (NCSs) have received increasing attention in recent years, due to their potential applications in transportation systems, cloud control systems, power grids, etc. Comparing with traditional point-to-point architecture, NCSs have many nice features which include but are not limited to simpler maintenance and installation and larger flexibility [1,2]. Due to continuous-time system modeling and practical physical plant as well as the sampling behaviour in NCSs, there exist both continuous-time and discrete-time signals in NCSs, and thus forming the so-called sampled-data NCSs. However, this hybrid nature of the sampled-data NCSs brings challenges in designing an effective control scheme and analyzing the close-loop system stability and performance, especially for constrained nonlinear systems, see [3–5]. Fortunately, a powerful way to tackle constrained nonlinear system is model predictive control (MPC), which solves a

finite horizon optimal control problem (FHOCP) online meanwhile taking the constraints into consideration. Consequently, many investigations on MPC for nonlinear NCSs have been reported, the associated works can be seen in [6–12] and the reference therein.

In general, MPC for sampled-data NCSs can be studied under two framework, i.e., the continuous-time framework [6–8] and the discrete-time framework [9–11]. On the one hand, the continuous-time framework is focused as the actual plants are usually modeled by using continuous-time ODEs subjected to continuous-time constraints. In continuous-time framework, these constraints in the FHOCP are considered in a continuous-time setting, and thus can be naturally satisfied in continuous-time sense. However, two non-negligible problems arise in continuous-time framework: (i) the continuous-time nonlinear ODEs are regarded as one constraint in the FHOCP, thus the computation procedure of the MPC is always supposed to be continuously repeated over a vanishingly small sampling time, which is computationally intractable, and (ii) stability of the continuous-time systems may no longer be maintained after applying the sample-and-hold control signal, see [13]. On the other hand, the discrete-time framework is widely discussed because MPC algorithms in this framework can be easily implemented in the digital platform and also have computational advantage [14,15]. Nonetheless, there still exist some problems in discrete-time framework as follows: (i) the state constraints are considered only at each sampling instant, thus the inter-sampling behaviour, i.e., the state evolutions between two

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consecutive sampling instants, is neglected, leading to continuous-time state constraints unsatisfaction; (ii) the required exact discrete-time model (DTM) in solving the FHOCPC may not be available for general nonlinear system; (iii) the stability for the exact DTM can be guaranteed, but the original continuous-time system can not due to the neglect of the inter-sampling behaviour.

Some preliminary results focusing on the above problems that occurred in discrete-time and continuous-time frameworks are reported in [16–19]. In [17], the sampled-data MPC for linear systems is studied. Since linear system is considered, whose exact discretized model is easily obtained and the inter-sample behavior can be explicitly analyzed, the computation challenge of the MPC is overcome under the discrete-time framework and the constraint satisfaction is achieved in continuous-time sense by employing the polytopic overapproximations technique. However, such technique does not work for nonlinear systems as the discretization of general nonlinear systems is generally intractable. To overcome this drawback, receding horizon control scheme for continuous-time nonlinear systems via their approximations is studied in [19], where the approximate DTM of nonlinear system is introduced to replace its exact DTM. But in these works, the state constraints that can be encountered in many practical control systems, have not been considered. Further studies can be found in [16,18], where state constraints are considered only at the sampling instants, but the inter-sampling behavior is neglected, thus the state constraints may be violated in the continuous-time sense. Additionally, none of these preliminary results study model error between the obtained approximate DTM and the original continuous-time system. Indeed, the combined effect of the model error and the disturbances will deteriorate the control performance, the worst case may cause infeasibility and instability.

With the above motivations, we investigate MPC for continuous-time perturbed nonlinear systems, considering two problems: (i) how to implement MPC for continuous-time systems under the digital environment without transgressing the continuous-time constraints? (ii) how to establish the stability of the original system? These two problems are quite essential for the implementation of MPC algorithms for sampled-data NCSs, and we thus develop a novel RAMPC scheme to solve them. The main contributions of our work are twofold:

- A novel constraint tightening approach for the formulation of the FHOCPC is proposed. This approach integrates the external disturbance and the model error between the approximate DTM and continuous-time systems, and consequently, a tightened set is defined to achieve continuous-time state constraints satisfaction. Compared with the existing work where the state constraints are either not considered or considered only at the sampling instants, the robust constraint satisfaction in continuous-time sense achieved in this work has more theoretical and practical significance.
- A new condition for guaranteeing feasibility and stability is established. By incorporating the upper bound of the model error and disturbance, this condition provides a guideline to determine the allowable sampling period when the disturbance is known or determine the maximum allowable disturbance bound when the sampling period is selected. Moreover, it is worth noting that we prove that the resultant closed-loop system is input-to-state practically stable (ISpS) under this new conditions.

The remainder of the paper is shown in five sections. Section 2 describes the research problem. The main results are shown in Section 3, including the specific RAMPC scheme, recursive feasibility as well as the associated stability. The effectiveness is verified by a numerical example in Section 4. Section 5 concludes this work.

Notations. Let \mathbb{R} and \mathbb{Z}_0 represent the real and nonnegative integers, respectively. \mathbb{R}^n denotes the n -dimensional Euclidean space. $\lambda_{\min}(R)$ is the minimum eigenvalues of the matrix R . A matrix P is called positive definite when $P > 0$. For a vector x , its Euclidean norm is denoted by $\|x\| := \sqrt{x^T x}$ and the P -weighted norm is denoted by $\|x\|_P := \sqrt{x^T P x}$. $\mathbf{w}_{[t_1, t_2]}$ is a signal from t_1 to t_2 and its subscript can be omitted for simplify when it can be derived from context. For two sets $A, B \subseteq \mathbb{R}^n$, $A \ominus B := \{x : x + y \in A, \forall y \in B\}$ denotes the Pontryagin difference set, $A \oplus B := \{x + y : x \in A, y \in B\}$ denotes the Minkowski addition set. \mathcal{K} denotes a class- \mathcal{K} , \mathcal{K}_∞ is a class- \mathcal{K}_∞ and \mathcal{KL} is a class- \mathcal{KL} functions, see [20] for details.

2. Problem formulation

Consider the following uncertain nonlinear control system

$$\dot{x}(t) = f(x(t), u(t)) + w(t), t \geq 0 \tag{1}$$

where the system state $x(t) \in \mathcal{X} \subseteq \mathbb{R}^n$ and control input $u(t) \in \mathcal{U} \subseteq \mathbb{R}^m$. The vector $w(t) \in \mathcal{W} = \{w(t) \in \mathbb{R}^n : \|w\|_R \leq \xi\} \subseteq \mathbb{R}^n$ is the external disturbance with a positive definite matrix R . These two compact sets \mathcal{X} and \mathcal{U} represent the continuous-time state and control input constraints sets that contain the origin. Let the solution of system (1) for time t with the control input \mathbf{u} , disturbance \mathbf{w} and initial value $x(0) = x_0$ be denoted by $\varphi(t; x_0, \mathbf{u}, \mathbf{w})$. Moreover, the function f with $f(0, 0) = 0$ should satisfy the following assumption.

Assumption 1. f is local Lipschitz continuous with a constant $L_f > 0$ depending on the weighted matrix R , i.e. $\|f(x, u) - f(y, u)\|_R \leq L_f \|x - y\|_R, \forall x, y \in \mathcal{X}$ and $u \in \mathcal{U}$.

In the implementation of MPC for sampled-data NCSs, the state $x(kT)$ is sampled at each sampling instant kT and is transmitted to MPC over a communication network, the FHOCPC is solved by exploiting this initial value $x(kT)$ to obtain the control input sequence. Then, the actual control, sent to the actuator with a zero-order holder over time interval $[kT, kT + T)$, is the first element in the above resultant control sequence. Finally, starting from the time instant $(k + 1)T$, the FHOCPC enters the next sampling and computation cycle.

Since the DTM is necessary in computation of the FHOCPC under digital environment, the original continuous-time model should be discretized. We first introduce some concepts about DTM of system (1). Given a sampling period T , then the exact DTM of system (1) with the sample-and-hold control input $u(t) = u(kT)$, if $t \in [kT, kT + T)$ is written as

$$x_{k+1} = x_k + \int_{kT}^{kT+T} (f(x(\tau), u_k) + w(\tau)) d\tau =: F_T^c(x_k, u_k) + w_{T,k} \tag{2}$$

where the simplified notions x_k and u_k are defined by $x_k := x(kT)$ and $u_k := u(kT), k \in \mathbb{Z}_0$, respectively, the subscript T means that F_T^c is parameterized with T , and $w_{T,k} := \int_{kT}^{kT+T} w(\tau) d\tau$ satisfying $w_{T,k} \in \mathbb{W} := \{w_T : \|w_T\|_R \leq \xi T\}$.

With the exact DTM (2), the corresponding FHOCPC under discrete-time framework is defined as follows:

$$\min_{\mathbf{u}_k} J(x_k, \hat{\mathbf{u}}_k, N)$$

- s.t. $\hat{x}_{k+i+1|k} = F_T^c(\hat{x}_{k+i|k}, \hat{u}_{k+i|k}),$
 $\hat{x}_{k+i|k} \in \mathcal{X},$
 $\hat{u}_{k+i|k} \in \mathcal{U},$
 $\hat{x}_{k+N|k} \in \mathcal{X}_f,$ (3)

where N is the prediction horizon, $i = 0, \dots, N-1$, $\hat{x}_{k+i|k}$ denotes the $(k+i)$ th predicted state based on the F_T^e and control sequence $\hat{\mathbf{u}}_k$ with initial state $\hat{x}_{k|k} = x_k$, the terminal state constraint set is \mathcal{X}_f , the cost function $J(x_k, \hat{\mathbf{u}}_k, N)$ is formulated as

$$J(x_k, \hat{\mathbf{u}}_k, N) = \sum_{i=0}^{N-1} l(\hat{x}_{k+i|k}, \hat{\mathbf{u}}_{k+i|k}) + g(\hat{x}_{k+N|k}) \quad (4)$$

with $l: \mathbb{R}^n \times \mathbb{R}^m \rightarrow \mathbb{R}^+$ and $g: \mathbb{R}^n \rightarrow \mathbb{R}^+$ being the stage cost function and the terminal cost, respectively.

Solving the FHOCPC based on (3) and (4), an optimal control sequence $\hat{\mathbf{u}}_{[k, k+N-1|k]}^0 = \{\hat{u}_{k|k}^0, \hat{u}_{k+1|k}^0, \dots, \hat{u}_{k+N-1|k}^0\}$ is obtained, then the actual control input that is applied to continuous-time system (1) is formulated as

$$u(t) = \kappa^{MPC}(x_k), t \in [kT, kT + T) \quad (5)$$

where $\kappa^{MPC}(x_k) := \hat{u}^0(k|k)$ called sample-and-hold signal, representing the MPC feedback control law, is the first element of $\hat{\mathbf{u}}_{[k, k+N-1|k]}^0$.

Although the above formulation gives a way to obtain a control sequence, some problems still exist and need further treatments:

- The exact DTM is general unavailable as solving the differential Eq. (1) with nonlinear term $f(x, u)$ is quite difficult;
- The obtained control sequence by solving FHOCPC (3) only guarantees the constraint satisfaction at the sampling instant. But when considering the inter-sampling behavior, the state constraints in continuous-time may be transgressed, i.e., $x(t) \notin \mathcal{X}$ for all $t \in (kT, kT + T)$.
- The MPC scheme guarantees that the exact DTM (2) is stable in some sense, but the stability of the continuous-time (1) with control signal (5) cannot be guaranteed.

Therefore, our objective is to design a scheme such that the above three problems are solved. To be specific, the control input obtained by solving a proper designed FHOCPC can stabilize the original continuous-time system (1) without resorting to the exact DTM (2) and violating state and control input constraints satisfaction in continuous-time.

Before proceeding, we introduce a necessary definition.

Definition 1 [21]. A set Φ is called robust positively invariant set for a uncertain system $x_{k+1} = F(x_k, w_{T,k})$ if for all $x_k \in \Phi$, then $F(x_k, w_{T,k}) \in \Phi$ for all $w_{T,k} \in \mathbb{W}$.

3. Main results

In this section, the RAMPC scheme is proposed, the recursive feasibility is analyzed and the associated stability results are established.

3.1. RAMPC scheme

In this part, we formulate the FHOCPC of the RAMPC algorithm, where the DTM, the state constraints, and the cost function are elaborated in the following.

Since the exact DTM F_T^e of system (1) is usually unavailable, we consider its approximate DTM instead. Let the sampling period T be given, the disturbance-free approximate DTM of system (1) is denoted by

$$x_{k+1} = F_T^a(x_k, u_k) \quad (6)$$

where the explicit expression of F_T^a depends on the adopted numerical methods, which include, but are not limited to, the Euler method and the modified Euler method. Taking the Euler approximation for example, we have $F_T^a(x_k, u_k) = F_T^{Euler}(x_k, u_k) := x_k + Tf(x_k, u_k)$.

Furthermore, F_T^a should satisfy the following assumptions.

Assumption 2. Given any sampling periods $T > 0$, F_T^a is continuous in u , the following two inequalities

$$\|F_T^a(x_1, u) - F_T^a(x_2, u)\|_R \leq e^{L_f T} \|x_1 - x_2\|_R \quad (7)$$

$$\|F_T^e(x_1, u) - F_T^a(x_1, u)\|_R \leq T\varrho(T) \quad (8)$$

hold with a \mathcal{K}_∞ function ϱ for all $x_1, x_2 \in \mathcal{X}$ and $u \in \mathcal{U}$.

Remark 1. The above assumption is very general. Inequality (7) implies that F_T^a satisfies a local Lipschitz condition. In fact, if the Euler approximation method is adopted, i.e., $F_T^a = F_T^{Euler}$, then based on Assumption 1, it can be easily observed that $\|F_T^a(x, u) - F_T^a(y, u)\|_R \leq (1 + L_f T)\|x - y\|_R$ and $1 + L_f T \leq e^{L_f T}$. As for the other approximate models, the corresponding analysis can be seen in [22]. Inequality (8) imposes a restriction on the approximation models by restricting the one step modeling error between F_T^a and F_T^e over the time interval $[kT, kT + T)$. It is necessary to point out that this inequality can be checked even though we do not know the exact DTM F_T^e , and more details can be found in [20,23].

To guarantee that state constraints are satisfied, the constraint tightening scheme is proposed. The principle of the tightening scheme is using the tightened state constraint set based on the upper bound of the state error between the original system (1) and the disturbance-free approximate DTM F_T^a .

Lemma 1. Suppose that Assumptions 1 and 2 hold. Define $t_1 = (k+i)T + \tau$, $i = 0, \dots, N-1$ with $\tau \in [0, T)$, then the state error $\mathbf{e}_{[(k+i)T, t_1]} := \varphi(t_1; x_{k+i}, \mathbf{u}_{[(k+i)T, t_1]}, \mathbf{w}_{[(k+i)T, t_1]}) - \hat{x}_{k+i|k}$ is bounded by

$$\|\mathbf{e}_{[(k+i)T, t_1]}\|_R \leq \mu\tau + \frac{e^{\mu T} - 1}{e^{L_f T} - 1} (T\varrho(T) + \zeta T) \quad (9)$$

for all $x_{k+i} \in \mathcal{X}$, $\mathbf{u}_{[(k+i)T, t_1]} \in \mathcal{U}$ and $\mathbf{w}_{[(k+i)T, t_1]} \in \mathcal{W}$, where μ is a constant that makes system function in (1) bounded from above, i.e., $\|f(x(t), u(t)) + w(t)\|_R \leq \mu, \forall x(t) \in \mathcal{X}, u(t) \in \mathcal{U}$, and $w(t) \in \mathcal{W}$.

Proof. Note that the constant μ always exists since the domain $\mathcal{X} \times \mathcal{U}$ is bounded and the external disturbance \mathcal{W} is a compact set. For better reading, the validity of this lemma is divided into three steps.

- First, let the state error between the exact DTM (2) and the approximate DTM (6) be defined as $e_{k+i} = x_{k+i} - \hat{x}_{k+i|k}$. By virtue of the definition of F_T^e, F_T^a and inequality (8), it comes that

$$\|F_T^e(x_k, u_k) + w_{T,k} - F_T^a(x_k, u_k)\|_R \leq T\varrho(T) + \zeta T$$

According to the Lipschitz property of f and F_T^a in Assumptions 1 and 2 and the triangle inequality, it is easy to obtain

$$\begin{aligned} \|e_{k+1}\|_R &\leq \|F_T^e(x_k, u_k) + w_{T,k} - F_T^a(x_{k|k}, u_k)\|_R \\ &+ \|F_T^a(x_{k|k}, u_k) - F_T^a(\hat{x}_{k|k}, u_k)\|_R \\ &= T\varrho(T) + \zeta T + e^{L_f T} \|x_k - \hat{x}_{k|k}\|_R \\ &= T\varrho(T) + \zeta T \end{aligned}$$

where the initial condition $\hat{x}_{k|k} = x_k$ and the fact $\hat{u}_{k|k} = u_k$ are utilized in these inequalities above. Then, it follows by induction that

$$\begin{aligned} \|e_{k+2}\|_R &\leq TQ(T) + \zeta T + e^{L_f T} \|x_{k+1} - \hat{x}_{k+1|k}\|_R \\ &\leq TQ(T) + \zeta T + e^{L_f T} (TQ(T) + \zeta T) \\ &\vdots \\ \|e_{k+i}\|_R &\leq \sum_{j=0}^{i-1} e^{jL_f T} (TQ(T) + \zeta T) \\ &= \frac{e^{iL_f T} - 1}{e^{L_f T} - 1} (TQ(T) + \zeta T) \end{aligned} \quad (10)$$

2. Secondly, we explain that the trajectory of the real system does not deviate far away from the approximate DTM over time interval $[(k+i)T, t_1]$. Since $\|f(x(t), u(t)) + w(t)\|_R \leq \mu$ holds and the solution of system (1) exists, it is easy to derive that $\|\varphi(t_1; x_{k+i}, \mathbf{u}_{[(k+i)T, t_1]}, \mathbf{w}_{[(k+i)T, t_1]}) - x_{k+i}\|_R \leq \mu\tau$.
3. Finally, incorporating the above two steps, we obtain

$$\begin{aligned} \|\mathbf{e}_{[(k+i)T, t_1]}\|_R &\leq \|\varphi(t_1; x_{k+i}, \mathbf{u}_{[(k+i)T, t_1]}, \mathbf{w}_{[(k+i)T, t_1]}) - x_{k+i}\|_R + \|e_{k+i}\|_R \\ &\leq \mu\tau + \frac{e^{iL_f T} - 1}{e^{L_f T} - 1} (TQ(T) + \zeta T) \end{aligned}$$

These complete the proof. \square

Based on Lemma 1, the tightened set is defined by the Pontryagin difference set

$$\mathcal{X}_i = \mathcal{X} \ominus \mathcal{B}_i \quad (11)$$

where the set \mathcal{B}_i is defined as

$$\mathcal{B}_i := \left\{ x \in \mathbb{R}^n : \|x\|_R \leq \mu\tau + \frac{e^{iL_f T} - 1}{e^{L_f T} - 1} (TQ(T) + \zeta T) \right\}.$$

One can note that if the predicted state satisfies the pointwise constraint $\hat{x}_{k+i|k} \in \mathcal{X}_i$, then the actual continuous-time system state constraints are satisfied, i.e., $\varphi(t_1; x_{k+i}, \mathbf{u}, \mathbf{w}) \in \mathcal{X}$. This claim is a direct result of the definition of the Pontryagin difference set and inequality (9). By this approach, the problem of the constraint satisfaction in continuous-time sense (the problem (b) in previous section) is solved.

With the above preliminaries, the FHOCP of the RAMPC scheme is formulated as follows. Given a sampling period T , the cost function related to T is defined by

$$J_T(x_k, \hat{\mathbf{u}}_k, N) = \sum_{i=0}^{N-1} l(\hat{x}_{k+i|k}, \hat{u}_{k+i|k}) + g(\hat{x}_{k+N|k}) \quad (12)$$

where $l(x, u) = \|x\|_Q^2 + \|u\|_P^2$ is the stage cost and $g(x) = \|x\|_R^2$ is the terminal cost with Q, P, R being positive definite matrices. The control sequence $\hat{\mathbf{u}}_{[k, k+N-1|k]}^0$ and the corresponding predictive state sequence $\hat{\mathbf{x}}_{[k, k+N|k]}^0$ are obtained by solving the FHOCP that is defined below

$$\min_{\mathbf{u}_k} J_T(x_k, \hat{\mathbf{u}}_k, N) \quad (13a)$$

$$s.t. \hat{x}_{k+i+1|k} = F_T^a(\hat{x}_{k+i|k}, \hat{u}_{k+i|k}), \quad (13b)$$

$$\hat{x}_{k+i|k} \in \mathcal{X}_i, \quad (13c)$$

$$\hat{u}_{k+i|k} \in \mathcal{U}, i = 0, \dots, N-1 \quad (13d)$$

$$\hat{x}_{k+N|k} \in \mathcal{X}_f,$$

with $\hat{x}_{k|k} = x_k$ being the initial condition of solving the FHOCP and $\mathcal{X}_f = \{x : \|x\|_R^2 \leq \sigma_T^2\}$ being the terminal state constraint set.

Remark 2. One may note that the upper bound of the state error in Lemma 1 may be over-conservative since the repeatedly use of the constant L_f that may be very large. Fortunately, this problem can be alleviated as we find a smaller L_f by changing another matrix-weighted norm, see [24] for more details.

Remark 3. Recall the three problems proposed in Section 2. The approximate DTM F_T^a is employed in (13a) to replace the exact DTM F_T^e , which solves the problem (a). To solve the problem (b), the tightened set in (13b) is proposed such that the state constraint is satisfied in continuous-time sense. As for the stability problem (c), since the cost function (4) that is independent on sampling period T may be unable to stabilize the exact DTM (2), typical examples can be found in [19,22], we employ a T -related cost function (12) here. Together with an additional assumption (Assumption 4), the stability for the continuous-time system can be established, which will be discussed in the next section. One may notice that with the development of wireless communication technique like 5G, low latency and high reliability can be achieved. Therefore, it is legitimate to not consider the delays and packet dropouts in this work.

The overall control structure with the RAMPC scheme is illustrated in Fig.1. At each sampling instant kT , the sampled state $x(kT)$ is transmitted to MPC over a communication network, the related FHOCP in (13) which takes $x(kT)$ as the initial condition is solved, and the control signal $\hat{u}_{k|k}$ selected from the obtained input sequence $\hat{\mathbf{u}}_{[k, k+N-1|k]}^0$ is sent to the actuator through a communication network again. The actuator provides the actual input $u(t) = \hat{u}_{k|k}$ over time interval $[kT, kT + T)$ to original system (1) in a sample-and-hold manner. The feasibility of this RAMPC scheme as well as the associated stability of the resultant overall system under some mild conditions are discussed in next section.

3.2. Recursive feasibility and stability

The feasibility of this RAMPC scheme as well as its associated stability are established in this part. The feasibility means the solution of the FHOCP always exists at each sampling instant. To guarantee feasibility and stability, some fairly standard assumptions are needed.

Assumption 3. The terminal state constraint set \mathcal{X}_f , the terminal cost $g(x)$, the auxiliary control law $h(x)$, another important set Ξ , and the stage cost function $l(x, u)$ satisfy the following properties,

1. $0 \in \mathcal{X}_f, \mathcal{X}_f \subset \Xi = \{x : \|x\|_R^2 \leq \bar{\sigma}_T^2\}$ with $0 < \sigma_T < \bar{\sigma}_T$ and $\Xi \subseteq \{x \in \mathcal{X}_{N-1} : h(x) \in \mathcal{U}\}$;
2. $F_T^a(x, h(x)) \in \mathcal{X}_f, \forall x \in \Xi$;
3. $g(F_T^a(x, h(x))) - g(x) \leq -l(x, h(x)), \forall x \in \Xi$;

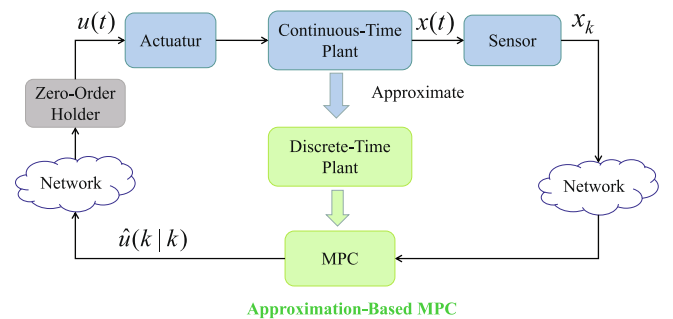


Fig. 1. The framework of the RAMPC scheme.

4. $g(x)$ is Lipschitz continuous in Ξ with constant $L_g > 0$ relying on weighted matrix R , i.e., $|g(x) - g(y)| \leq L_g \|x - y\|_R, \forall x, y \in \Xi$;
5. $g(x)$ is bounded, specifically, $\tilde{\alpha}_1(\|x\|) \leq g(x) \leq \tilde{\alpha}_2(\|x\|)$ with $\tilde{\alpha}_1, \tilde{\alpha}_2 \in \mathcal{K}_\infty, \forall x \in \Xi$;
6. $l(x, u)$ is Lipschitz continuous with $L_l \geq 0$ depending on weighted matrix R , i.e., the inequality $|l(x, u) - l(y, u)| \leq L_l \|x - y\|_R$ holds $\forall x, y \in \mathcal{X}, u \in \mathcal{U}$;
7. $l(x, u) \geq \tilde{\alpha}_3(\|x\|)$ with $\tilde{\alpha}_3 \in \mathcal{K}_\infty, \forall x \in \mathcal{X}, u \in \mathcal{U}$.

Assumption 4. ([18]) $f(x, u)$ in (1) is bounded from above, i.e., the following inequality hold with $\varrho_1 \in \mathcal{K}_\infty$

$$\|f(x, u)\| \leq \max\{\theta, \varrho_1(1/\theta)l(x, u)\} \quad (14)$$

for all $\theta > 0, x \in \mathcal{X}$ and $u \in \mathcal{U}$.

Remark 4. The Properties 1–7 in Assumption 3 provides some rules in designing the matrices P, Q, R , the set Ξ and the auxiliary controller $h(x)$, which are fairly standard in the field of MPC, see, e.g., [24,14]. It should be noted that the parameters σ_T and $\tilde{\sigma}_T$ are both dependent on the sampling period T , which is a immediate result of the Properties 2–3. Assumption 4 can be easily satisfied for very general continuous function $f(x, u)$, see [18, Section 4.3] for more details.

First of all, the following theorem formulates the upper bound of the model error and disturbance to guarantee that the FHOCP is recursively feasible.

Theorem 1. Given the states set $\bar{\mathcal{X}}^{MPC}$ and suppose that the FHOCP in (13) has a solution for any $x_0 \in \bar{\mathcal{X}}^{MPC}$, if the system (1) fulfills Assumption 1–3, the model error and disturbance satisfy

$$\xi T + T\varrho(T) \leq \frac{\tilde{\sigma}_T - \sigma_T}{e^{(N-1)L_f T}} \quad (15)$$

then $\bar{\mathcal{X}}^{MPC}$ is robust positively invariant for the exact DTM (2).

Proof. To prove this claim, one only needs to prove that if $x_k \in \bar{\mathcal{X}}^{MPC}$, then $x_{k+1} \in \bar{\mathcal{X}}^{MPC}$.

The proof of the robust invariance of $\bar{\mathcal{X}}^{MPC}$ can be completed by induction. Assume that the FHOCP has a the optimal solution $\hat{u}_{[k, k+N-1|k]}^0$ at kT , then based on which a solution of the FHOCP at time $(k+1)T$ can be constructed as follows, and we only need to verify its feasibility.

$$\hat{u}_{k+1+i|k+1} = \begin{cases} \hat{u}_{k+1+i|k}^0 & i = 0, \dots, N-2 \\ h(\hat{x}_{k+N|k+1}) & i = N-1 \end{cases} \quad (16)$$

For clarity of exposition, the proof is divided into three steps.

- $\hat{x}_{k+1+N|k+1} \in \mathcal{X}_f$: Applying the control input $\hat{u}_{k+1+i|k+1} = \hat{u}_{k+1+i|k}^0$ to the approximate DTM, then the state error in the following is bounded by

$$\begin{aligned} & \|\hat{x}_{k+1+i|k+1} - \hat{x}_{k+1+i|k}^0\|_R \\ &= \|F_T^a(\hat{x}_{k+i|k+1}, \hat{u}_{k+i|k}^0) - F_T^a(\hat{x}_{k+i|k}^0, \hat{u}_{k+i|k}^0)\|_R \\ &\leq e^{L_f T} \|\hat{x}_{k+i|k+1} - \hat{x}_{k+i|k}^0\|_R \\ &\quad \dots \\ &\leq e^{iL_f T} \|\hat{x}_{k+1|k+1} - \hat{x}_{k+1|k}^0\|_R \end{aligned} \quad (17)$$

where the inequality (7) in Assumption 2 is used. Noticing that $\hat{x}_{k+1|k+1} = x_{k+1}$, thus from (10), we have $x_{k+1} - \hat{x}_{k+1|k}^0 = T\varrho(T) + \xi T$. Substituting it into (17) yields

$$\|\hat{x}_{k+1+i|k+1} - \hat{x}_{k+1+i|k}^0\|_R \leq e^{iL_f T} [T\varrho(T) + \xi T] \quad (18)$$

Since $\xi T + T\varrho(T) \leq \frac{\tilde{\sigma}_T - \sigma_T}{e^{(N-1)L_f T}}$. Thus, we have

$$\|\hat{x}_{k+N|k+1}\|_R \leq \|\hat{x}_{k+N|k}^0\|_R + \tilde{\sigma}_T - \sigma_T$$

Noting that $\hat{x}_{k+N|k}^0 \in \mathcal{X}_f$, i.e., $\|\hat{x}_{k+N|k}^0\|_R \leq \sigma_T$, we have

$$\|\hat{x}_{k+N|k+1}\|_R \leq \tilde{\sigma}_T \quad (19)$$

which means $\hat{x}_{k+N|k+1} \in \Xi$. Finally, utilizing the auxiliary controller $h(\hat{x}_{k+N|k+1})$ and considering the Property 2 in Assumption 3, one has $\hat{x}_{k+1+N|k+1} \in \mathcal{X}_f$.

- $\hat{x}_{k+1+i|k+1} \in \mathcal{X}_i$: First, for $i = 0, \dots, N-2$, considering the formulation in (18), we obtain

$$\|\hat{x}_{k+1+i|k+1}\|_R \leq \|\hat{x}_{k+1+i|k}^0\|_R + e^{iL_f T} [T\varrho(T) + \xi T]$$

Since $\hat{x}_{k+1+i|k}^0 \in \mathcal{X}_{i+1}$, we have

$$\begin{aligned} \hat{x}_{k+1+i|k+1} &\in \mathcal{X}_{i+1} \oplus e^{iL_f T} [T\varrho(T) + \xi T] \\ &\in \mathcal{X}_i \end{aligned}$$

Next, for $i = N-1$. From (19), we know that $\hat{x}_{k+N|k+1} \in \Xi$. Considering the Property 1 in Assumption 3, one has $\hat{x}_{k+N|k+1} \in \Xi \subseteq \mathcal{X}_{N-1}$.

- $\hat{u}_{k+1+i|k+1} \in \mathcal{U}$: Based on equation (16), $\hat{u}_{k+1+i|k+1} = \hat{u}_{k+1+i|k}^0 \in \mathcal{U}$ for $i = 0, \dots, N-2$ due to the feasibility of $\hat{u}_{i|k}^0$. Then, $\hat{x}_{k+N|k+1} \in \Xi$ as verified above, we have $h(\hat{x}_{k+N|k+1}) \in \mathcal{U}$ by virtue of the Property 3 in Assumption 3 and therefore, $\hat{u}_{k+N|k+1} \in \mathcal{U}$.

Based on the above three claims, the constructed solution (16) is feasible at time $(k+1)T$, which implies that $\bar{\mathcal{X}}^{MPC}$ is a robust positively invariant set, and thus this proof is completed.

Remark 5.

1) Note that the robust positive invariance of $\bar{\mathcal{X}}^{MPC}$ implies that the state x_k at every sampling instant is in $\bar{\mathcal{X}}^{MPC}$, which also means that the solution of the FHOCP (13) always exists, i.e., the RAMPC scheme is recursively feasible.

2) Inequality (15) provides a method to determine the sampling period T once we know the upper bound of the external disturbance in practice or to determine the maximum allowable disturbance bound ξ when the sampling period is selected. If we choose an approximate T^* to let

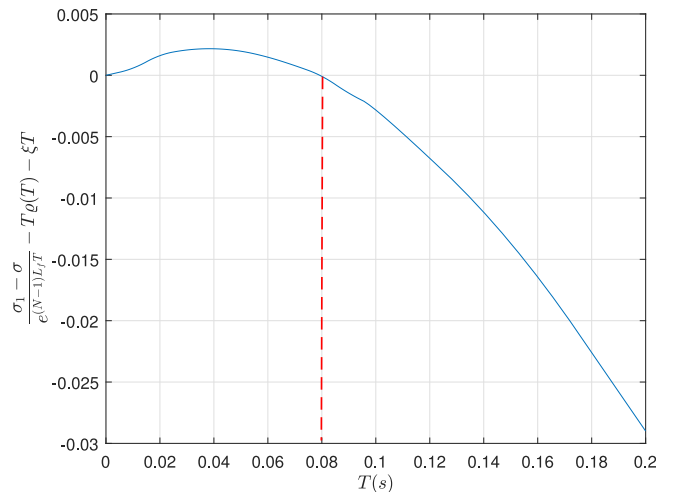


Fig. 2. The value of the function $G(T) = \frac{\tilde{\sigma}_T - \sigma_T}{e^{(N-1)L_f T}} - T\varrho(T) - \xi T$.

$\xi T^* + T^* \varrho(T^*) = \frac{\bar{\sigma}_{T^*} - \sigma_{T^*}}{e^{(\bar{n}-1)L_f T^*}}$ hold, then the condition (15) always holds for any $T \in (0, T^*]$. This claim is verified by numerical simulation in Section 4. Further, note that the maximum allowable disturbance bound ξ cannot be infinite since the right hand side of (15) is finite even if the sampling period T is sufficiently small (as illustrated in Fig. 2).

Theorem 2. Suppose that the sampling period T is selected to satisfy (15) and Assumptions 1–4 hold true. Then the exact DTM (2) under the RAMPC algorithm is ISpS for all $x_0 \in \bar{\mathcal{X}}^{\text{MPC}}$, i.e., the following inequality holds

$$\|x_k\| \leq \bar{\beta}(\|x_0\|, kT) + \bar{\gamma}(\xi) + \bar{\delta} \quad (20)$$

with two functions $\bar{\beta} \in \mathcal{KL}$, $\bar{\gamma} \in \mathcal{K}_\infty$ and a constant $\bar{\delta} > 0$ for all initial condition $x_0 \in \bar{\mathcal{X}}^{\text{MPC}}$ and disturbances $w_{T,k} \in \mathbb{W}$.

Proof. See Appendix. \square

The above theorem shows that the exact DTM (2) is stable with the control input provided by the RAMPC algorithm. On the other hand, Assumption 4 guarantees that the continuous-time solution of the original system (1) between the two sampling instants kT and $kT + T$ does not diverge far away from the system state at the sampling instant kT . Incorporating these two facts, the stability of the system (1) by applying control signal (5) is established.

Theorem 3. Suppose that Assumptions 1–4 and inequality (15) are satisfied, then the original system (1) with the sample-and-hold control signal (5) is ISpS, i.e., there are $\beta \in \mathcal{KL}$, $\gamma \in \mathcal{K}_\infty$ and a constant $\delta > 0$ such that

$$\|x(t)\| \leq \beta(\|x_0\|, t) + \gamma(\xi) + \delta \quad (21)$$

holds for any initial condition $x_0 \in \mathcal{X}^{\text{MPC}}$ and $w(t) \in \mathcal{W}^{\text{MPC}}$, where \mathcal{X}^{MPC} and \mathcal{W}^{MPC} are allowable state set and disturbance set defined below.

Before proceeding, we introduce a lemma which bounds the continuous-time solution between two sampling instants kT and $kT + T$ by the state at the sampling instant kT .

Lemma 2. If Assumptions 1–4 and inequality (15) hold, then there exist $\gamma_1, \gamma_2 \in \mathcal{K}_\infty$ such that the continuous-time solutions at time t satisfy

$$\|\varphi(t; x_0, \mathbf{u}, \mathbf{w})\| \leq \gamma_1(\|x_0\|) + \gamma_2(\xi) \quad (22)$$

for all $x_0 \in \bar{\mathcal{X}}^{\text{MPC}}$, $w(t) \in \mathcal{W}$ and $t \in [0, T]$.

Proof. For $t \in [0, T]$, leveraging inequality (14) to continuous-time solution yields

$$\begin{aligned} \|\varphi(t; x_0, \mathbf{u}, \mathbf{w}) - x_0\| &\leq \int_0^t \|f(x(\tau), u(\tau)) + w(\tau)\| d\tau \\ &\leq \frac{\xi T}{\lambda_{\min}(R)} + \max \left\{ T\theta, \varrho_1(1/\theta) \int_0^t l(x(\tau), u(\tau)) d\tau \right\} \end{aligned}$$

Notice that the inequality $l(x(t), u(t)) \leq ml(x_0, u_0)$ always holds with a constant $m > 0$ for $t \in [0, T]$ due to \mathcal{X}, \mathcal{U} are compact set and the Property 6 in Assumption 3, we obtain

$$\|\varphi(t; x_0, \mathbf{u}, \mathbf{w}) - x_0\| \leq \frac{\xi T}{\lambda_{\min}(R)} + \max \{ T\theta, m\varrho_1(1/\theta)Tl(x_0, u_0) \}$$

From the definition of $J_T^0(x(k))$ and inequality (25), one has $Tl(x_0, u_0) \leq J_T^0(x_0) \leq \alpha_2(\|x_0\|)$ for all $x_0 \in \bar{\mathcal{X}}^{\text{MPC}}$. Thus, we have

$$\|\varphi(t; x_0, \mathbf{u}, \mathbf{w}) - x_0\| \leq \max \{ T\theta, m\varrho_1(1/\theta)\alpha_2\|x_0\| \} + \frac{\xi T}{\lambda_{\min}(R)}$$

The remainder part is to construct two function $\gamma_1 \in \mathcal{K}_\infty$ and $\gamma_2 \in \mathcal{K}_\infty$, which has the similar outline as the one in [18, Remark 4.13]. Specifically, θ can be given by $\theta = \tilde{\gamma}(\|x_0\|)$, where

$$\tilde{\gamma}(r) := \begin{cases} 1/\varrho_1^{-1}(1/\sqrt{\alpha_2(r)}) & \text{if } r > 0 \\ 0 & \text{if } r = 0 \end{cases}$$

It is easily verified that $\tilde{\gamma} \in \mathcal{K}_\infty$. Hence, by letting

$$\gamma_1(r) = r + \max \{ T\tilde{\gamma}(r), m\sqrt{\alpha_2(r)} \}, \gamma_2(r) = \frac{Tr}{\lambda_{\min}(R)},$$

one finally obtains

$$\|\varphi(t; x_0, \mathbf{u}, \mathbf{w})\| \leq \gamma_1(\|x_0\|) + \gamma_2(\xi)$$

which complete this proof. \square

The proof of Theorem 3: On the basis of Theorem 2 and Lemma 2, the stability result can be established. Our objective is to find $\beta \in \mathcal{KL}$, $\gamma \in \mathcal{K}$ and $\delta > 0$ to make the inequality (21) hold. The following process follows the same line of Theorem 5 in [25]. Therefore, one only needs to pay attention to the set $\mathcal{X}^{\text{MPC}}, \mathcal{W}^{\text{MPC}}$ and expressions of $\beta(\cdot, \cdot), \gamma(\cdot)$ and δ . For simplicity, we directly give these expressions as follows:

$$\begin{aligned} \beta(s, \tau) &:= \max \left\{ \gamma_1(s)e^{T-\tau}, 2^{nT} \max_{\eta \in [0, \tau]} 2^{-\eta} \gamma_1(6\bar{\beta}(2\gamma_1(s), \tau - \eta)) \right\}, \\ \gamma(s) &:= \gamma_1(6\bar{\beta}(2\gamma_2(s), 0)) + \gamma_1(3\bar{\gamma}(s)) + \gamma_2(s), \\ \delta &:= \gamma_1(3\bar{\delta}) \end{aligned}$$

where n is a positive constant such that $nT \in \mathbb{Z}_0$ and $nT \geq 2$. Moreover, the compact sets \mathcal{X}^{MPC} and \mathcal{W}^{MPC} are taken as

$$\begin{aligned} \mathcal{X}^{\text{MPC}} &:= \left\{ x \in \mathbb{R}^n : x = \min \left\{ \gamma_1^{-1} \left(\frac{(1-a)\bar{x}}{2} \right), \gamma_1^{-1} \left(\bar{\beta}^{-1} \left(\frac{(1-a)\bar{x}}{2}, 0 \right) \right) \right\}, \forall \bar{x} \in \bar{\mathcal{X}}^{\text{MPC}} \right\} \\ \mathcal{W}^{\text{MPC}} &:= \left\{ \xi \in \mathbb{R}^n : \xi = \min \left\{ \gamma_2^{-1} \left(\frac{a\bar{\xi}}{2} \right), \bar{\gamma}^{-1} \left(\frac{a\bar{\xi}}{2} \right), \bar{\xi} \right\}, \forall \bar{x} \in \bar{\mathcal{X}}^{\text{MPC}}, \bar{\xi} \leq \frac{\bar{\sigma}_T - \sigma_T}{T\bar{\sigma}^{(n-1)L_f T}} - \rho(T) \right\} \end{aligned}$$

where a is an arbitrary number $0 < a < 1$.

These complete the proof.

4. Simulation example

Consider the following cart-damper-spring system discussed in [26]

$$\begin{cases} \dot{x}(t) = v(t) \\ \dot{v}(t) = -\frac{k_0}{M_c} e^{-x(t)} x(t) - \frac{h}{M_c} v(t) + \frac{u(t)}{M_c} + w(t), \end{cases} \quad (23)$$

where $x(t)$ is the cart displacement, $v(t)$ denotes the velocity, $M_c = 1.25$ kg is the cart mass, the other parameters are given as $k_0 = 0.9$ N/m, $h = 0.42$ N.s/m. $u(t)$ and $w(t)$ are respectively, the control input and the additive disturbance. Let $z = [x, v]$ denote the system states. The state and control constraints are set as $\mathcal{X} = \{z : -0.5 \leq z(t) \leq 0.5\}$ and $\mathcal{U} = \{u : -0.5 \leq u(t) \leq 0.5\}$. For simplify, we here formulate the specific form of the disturbance-free Euler approximation of the system (23).

$$\begin{cases} x_{k+1} = x_k + T v_k \\ v_{k+1} = \left(1 - \frac{Th}{M_c} \right) v_k - \frac{k_0 T}{M_c} e^{-x_k} x_k + \frac{T u_k}{M_c} \end{cases} \quad (24)$$

Note that the form of modified Euler approximation [27] and fourth-order Runge–Kutta approximation [28] can be similarly obtained.

Considering the MPC implementation of DTM (24), setting prediction horizon $N = 5$. The local Lipschitz constant is $L_f = 1.8440$. The two weighted matrices are set as $Q = [0.5 \ 0; 0 \ 0.5]$ and $P = 0.5$. For Euler approximation, the upper bound of one step modeling error between F_T^e and F_T^a can be obtained as $T\varrho(T) = L_f \mu T^2/2$. To determine a proper sampling period T , the

value of the function $G(T) = \frac{\sigma_T - \sigma_T}{e^{(N-1)LT} - TQ(T) - \xi T$ is depicted in Fig. 2 with external disturbance $\|w(t)\| \leq 0.005$. Then according to the condition (15) in Theorem 1, the sampling period can be set as $T = 0.08$ sec to guarantee the feasibility. The weighted matrix R then can be calculated as

$$R = \begin{bmatrix} 1.5385 & 0.5068 \\ 0.5068 & 0.9369 \end{bmatrix},$$

the terminal set $\mathcal{X}_f = \{z : \|z\|_R^2 \leq 0.3160^2\}$ and another important set $\Xi = \{z : \|z\|_R^2 \leq 0.3255^2\}$ can be determined by using the Jacobian linearization methods in [15].

To examine this scheme, we use MATLAB subroutine `fmincon` to solve the FHOCP. The initial value is $x_0 = [0.3, -0.3]$, the MPC algorithms based on the Euler approximation, modified Euler approximation and fourth-order Runge–Kutta approximation are performed, respectively, and the results are illustrated in Figs. 3–5. It is easy to observe that the continuous-time state and control constraints are all satisfied, and the overall system is ISpS. It is worth noting that the Runge–Kutta approximation is more accurate but always has a more complex expression, which increases

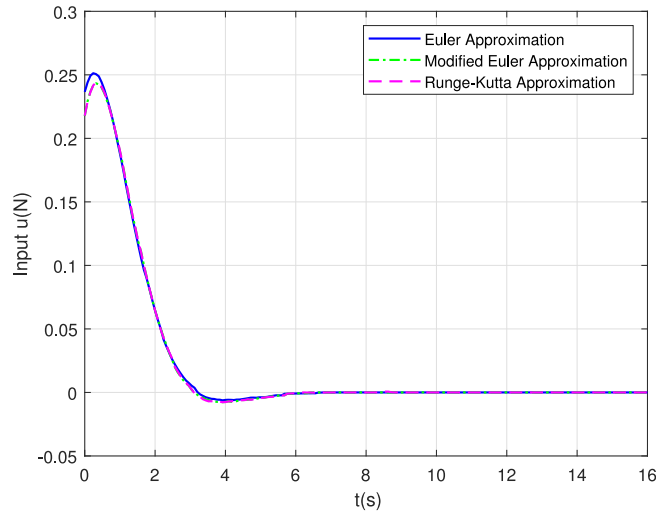


Fig. 5. Comparison of control inputs.

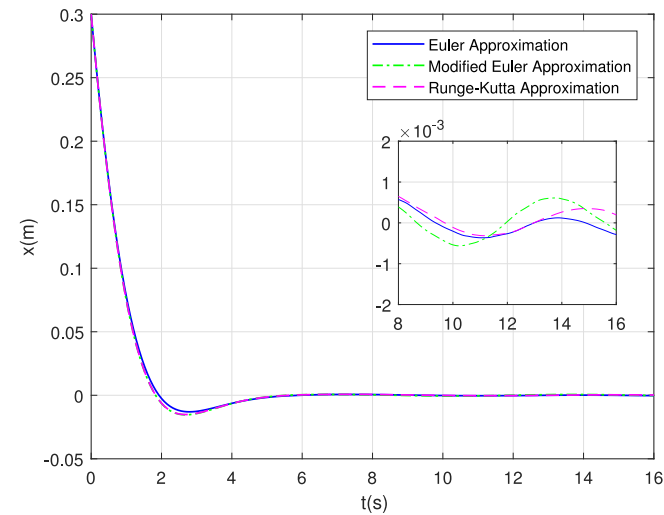


Fig. 3. Displacements of the closed-loop systems with different approximations.

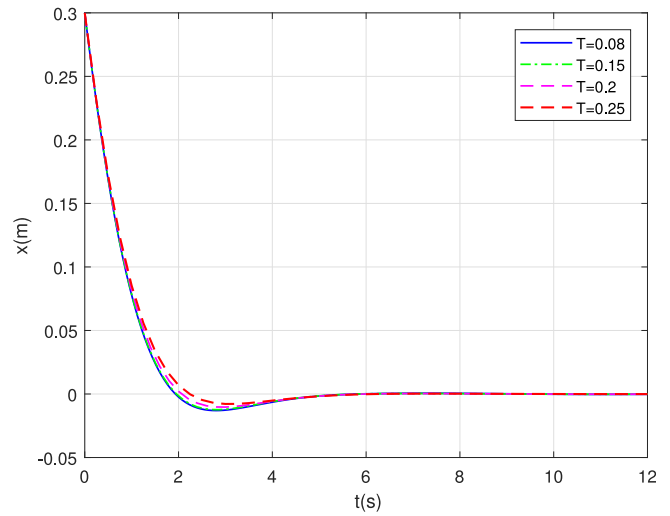


Fig. 6. Comparison of displacements with different sampling periods.

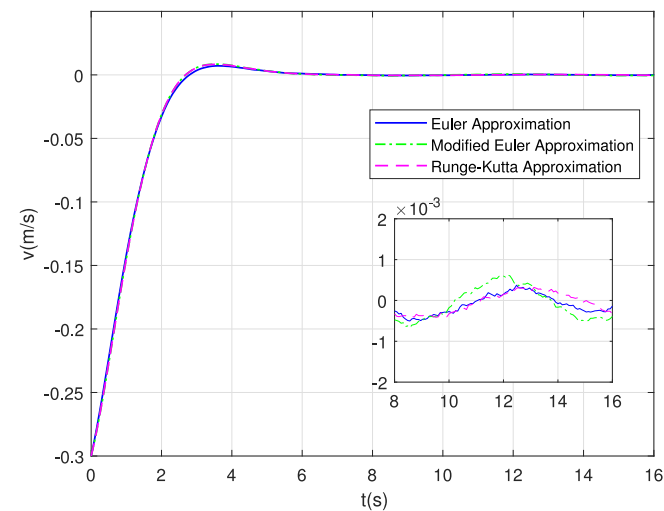


Fig. 4. Velocities of the closed-loop systems with different approximations.

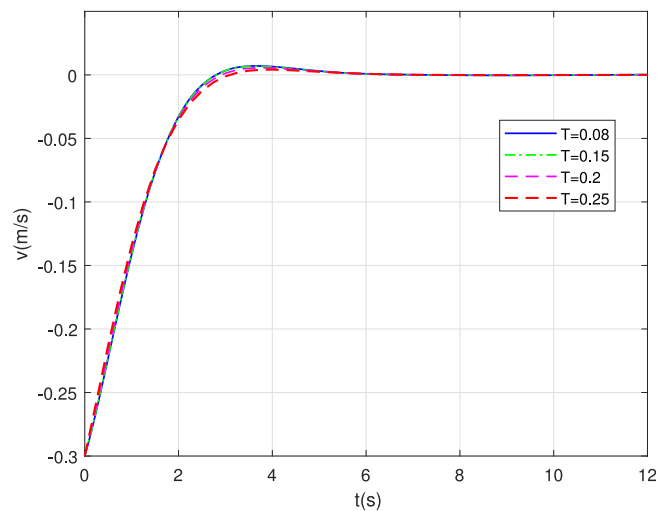


Fig. 7. Comparison of velocities with different sampling periods.

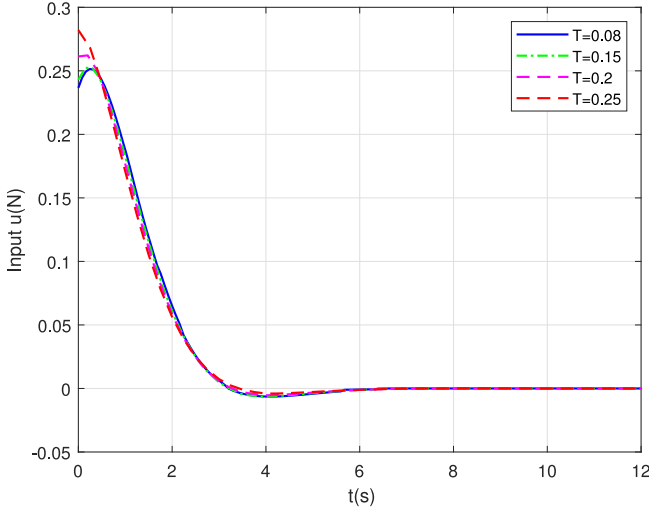


Fig. 8. Comparison of control inputs with different sampling periods.

difficulty in designing MPC algorithms. On the contrary, the Euler approximation is low-accuracy, but it is simpler in design and can obtain a comparable control performance under an appropriate sampling period.

Additionally, the effect of sampling period on the FHOCP is investigated by trying different sampling periods T for Euler approximation system (24). The resulting Figs. 6–8 show that the FHOCP is feasible under a small sampling period T . However, the FHOCP does not have feasible solutions in simulation after continuing to increase the sampling period, although the disturbance has been neglected. One can also observe that the FHOCP is still feasible even if the condition (15) is violated, this is because of the conservativeness brought by the Lipschitz constants and the constraint tightening technique [24]. Therefore, how to reduce such conservativeness is our future work.

5. Conclusion

The implementation of MPC for continuous-time systems in the digital environment has been investigated in this paper. A RAMPC scheme for NCSs has been proposed accordingly. With this scheme, the FHOCP can be solved in discrete-time manner, which makes the MPC algorithms computationally tractable. In addition, the continuous-time state and control constraints satisfaction have been achieved by a novel constraint tightening scheme. More importantly, we have shown that the FHOCP is recursively feasible at each sampling instant and the original continuous-time system is ISpS by applying the sample-and-hold control signal under some mild assumptions. Our future work is to reduce the conservativeness and the computational complexity of the FHOCP by considering the event-triggered MPC [29] and adaptive horizon MPC [14].

Declaration of Competing Interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

Appendix A. Proof of Theorem 2

Proof. According to [30], we can choose a suitable ISpS-Lyapunov function for the system (2) to prove that it is stable. We here take the optimal value function as the ISpS-Lyapunov function candi-

date. Let $J_T^0(x_k) = J_T(x_k^0, \hat{u}_k^0, N)$ be the optimal value function at the sampling instant kT . The following derivation we will show that the following

$$\begin{aligned} \alpha_1(\|x_k\|) &\leq J_T^0(x_k) \leq \alpha_2(\|x_k\|) \\ J_T^0(x_{k+1}) - J_T^0(x_k) &\leq T[-\alpha_3(\|x_k\|) + \tilde{\gamma}(\xi) + \delta_1] \end{aligned}$$

hold with functions $\alpha_1, \alpha_2, \alpha_3 \in \mathcal{K}_\infty$ and $\tilde{\gamma} \in \mathcal{K}$ for all $x_k \in \bar{\mathcal{X}}^{MPC}$, $w_{T,k} \in \mathbb{W}$.

Suppose that Assumption 3 hold, according to [16, Section 2.4], it is easy to obtain

$$J_T^0(x_k) \leq g(x_k) \leq \tilde{\alpha}_2(\|x_k\|), \quad \forall x_k \in \Xi$$

For the upper bound of $J_T^0(x_k)$ in $\bar{\mathcal{X}}^{MPC}$, a feasible method in [31] is utilized. Since the compactness of \mathcal{X}, \mathcal{U} and the fact of Assumption 3, there exists an upper bound for $J_T^0(x_k)$ such that $J_T^0(x_k) \leq \tilde{J}$ for all $x_k \in \bar{\mathcal{X}}^{MPC}$, where \tilde{J} is finite. Define a set $B_r \in \mathbb{R}^n, B_r = \{x \in \mathbb{R}^n : \|x\| \leq r\} \subset \mathcal{X}_f$. This set always exists since \mathcal{X}_f contains the origin. Let $\epsilon = \max\left(1, \frac{\tilde{J}}{\alpha_2(r)}\right)$ and $\alpha_2(\|x\|) = \epsilon \cdot \tilde{\alpha}_2(\|x\|)$, we have

$$J_T^0(x_k) \leq \alpha_2(\|x_k\|), \quad \forall x_k \in \bar{\mathcal{X}}^{MPC} \quad (25)$$

To get a lower bound for the optimal value function $J_T^0(x_k)$, for any $x_k \in \bar{\mathcal{X}}^{MPC} \setminus \{0\}$, if the corresponding optimal state evolution is denoted as $\hat{x}_{k+i+1|k}^0 = F_T^a(\hat{x}_{k+i|k}^0, \hat{u}_{k+i|k}^0)$, then we study two cases:

- 1) If $\|\hat{x}_{k+N|k}^0 - x_k\| \leq \frac{1}{2}\|x_k\|$, which implies $\|\hat{x}_{k+N|k}^0\| \geq \frac{1}{2}\|x_k\|$, according to the property 5 in Assumption 3, we have

$$J_T^0(x_k) \geq g(\hat{x}_{k+N|k}^0) \geq \tilde{\alpha}_1\left(\frac{1}{2}\|x_k\|\right)$$

- 2) If $\|\hat{x}_{k+N|k}^0 - x_k\| \geq \frac{1}{2}\|x_k\|$, then

$$\begin{aligned} \frac{1}{2}\|x_k\| &\leq \|\hat{x}_{k+N|k}^0 - x_k\| \\ &= \left\| \sum_{i=0}^{N-1} (\hat{x}_{k+1+i|k}^0 - \hat{x}_{k+i|k}^0) \right\| \\ &\leq \sum_{i=0}^{N-1} \|F_T^a(\hat{x}_{k+i|k}^0, \hat{u}_{k+i|k}^0) - \hat{x}_{k+i|k}^0\| \end{aligned} \quad (26)$$

where

$$\begin{aligned} \|F_T^a(x, u) - x\| &= \|F_T^a(x, u) - F_T^{Euler}(x, u) + Tf(x, u)\| \\ &\leq \frac{1}{\lambda_{\min}(R)} \|F_T^a(x, u) - F_T^{Euler}(x, u)\|_R + \|Tf(x, u)\| \end{aligned} \quad (27)$$

Let $\varrho_2 \in \mathcal{K}_\infty$ that make one step modeling error in (8) between the exact DTM F_T^a and the Euler approximate DTM F_T^{Euler} satisfy $\|F_T^a(x, u) - F_T^{Euler}(x, u)\|_R \leq T\varrho_2(T)$ and $\bar{\varrho} = \varrho + \varrho_2$, then

$$\|F_T^a(x, u) - F_T^{Euler}(x, u)\|_R \leq T\varrho(T) + T\varrho_2(T) = T\bar{\varrho}(T) \quad (28)$$

Substituting (27), (28) with $x = \hat{x}_{k+i|k}^0$ and $u = \hat{u}_{k+i|k}^0$ into (26) yields

$$\begin{aligned} \frac{1}{2}\|x_k\| &\leq \sum_{i=0}^{N-1} \left(\frac{1}{\lambda_{\min}(R)} T\bar{\varrho}(T) + T\|f(\hat{x}_{k+i|k}^0, \hat{u}_{k+i|k}^0)\| \right) \\ &\leq \sum_{i=0}^{N-1} \left(\max\{T\theta, \varrho_1(1/\theta)\Pi(\hat{x}_{k+i|k}^0, \hat{u}_{k+i|k}^0)\} + \frac{1}{\lambda_{\min}(R)} T\bar{\varrho}(T) \right) \\ &\leq \sum_{i=0}^{N-1} \left(\theta T + \varrho_1(1/\theta)\Pi(\hat{x}_{k+i|k}^0, \hat{u}_{k+i|k}^0) + \frac{1}{\lambda_{\min}(R)} T\bar{\varrho}(T) \right) \\ &\leq \frac{1}{\lambda_{\min}(R)} NT\bar{\varrho}(T) + N\theta T + \varrho_1(1/\theta)J_T^0(x_k) \end{aligned}$$

where inequality (14) is used in the third inequality. By choosing appropriate T^* to let $\frac{NT^* \bar{\rho}(T^*)}{\lambda_{\min}(R)} + N\epsilon T^* \leq \frac{1}{3} \|\mathbf{x}_k\|$, thus $J_T^0(\mathbf{x}_k) \geq \frac{1}{6\bar{\rho}_2(1/\epsilon)} \|\mathbf{x}_k\|$. Let $\alpha_1(\mathbf{x}) = \min\left\{\tilde{\alpha}_1\left(\frac{1}{2}\mathbf{x}\right), \frac{1}{6\bar{\rho}_2(1/\epsilon)}\mathbf{x}\right\}$, we have $J_T^0(\mathbf{x}_k) \geq \alpha_1(\|\mathbf{x}_k\|), \forall \mathbf{x}_k \in \bar{\mathcal{X}}^{\text{MPC}} \setminus \{0\}$. Note that this inequality also holds for $\mathbf{x}_k = 0$, thus the lower bound is established, i.e.,

$$J_T^0(\mathbf{x}_k) \geq \alpha_1(\|\mathbf{x}_k\|), \quad \forall \mathbf{x}_k \in \bar{\mathcal{X}}^{\text{MPC}} \quad (29)$$

Finally, the difference of the MPC cost function $\Delta J_k = J_T^0(\mathbf{x}_{k+1}) - J_T^0(\mathbf{x}_k)$ is calculated. For the sake of simplicity, using $\hat{\mathbf{x}}_{k+i}$ to represent $\hat{\mathbf{x}}_{k+1+i|k+1}$, $\hat{\mathbf{u}}_{k+i}$ to represent $\hat{\mathbf{u}}_{k+1+i|k+1}$, $\hat{\mathbf{x}}_{k+i+1}^0$ to denote $\hat{\mathbf{x}}_{k+i+1|k}^0$, $\hat{\mathbf{u}}_{k+i+1}^0$ to denote $\hat{\mathbf{u}}_{k+i+1|k}^0$. Since the feasible solution (16) which constructed in Theorem 1 may not optimal for the FHOCPC at time $(k+1)T$, and let $\bar{J}_T(\mathbf{x}_{k+1}) = J_T(\mathbf{x}_{k+1}, \hat{\mathbf{u}}_{k+1}, N)$, it follows

$$\begin{aligned} \Delta J_k &\leq \bar{J}_T(\mathbf{x}_{k+1}) - J_T^0(\mathbf{x}_k) \\ &= Tl(\hat{\mathbf{x}}_{k+N-1}, \hat{\mathbf{u}}_{k+N-1}) - Tl(\hat{\mathbf{x}}_k^0, \hat{\mathbf{u}}_k^0) \\ &\quad + \sum_{i=0}^{N-2} [Tl(\hat{\mathbf{x}}_{k+i}, \hat{\mathbf{u}}_{k+i}) - Tl(\hat{\mathbf{x}}_{k+i+1}^0, \hat{\mathbf{u}}_{k+i+1}^0)] \\ &\quad + g(\hat{\mathbf{x}}_{k+N}) - g(\hat{\mathbf{x}}_{k+N}^0) \end{aligned}$$

Adding and removing the same terms, it comes that

$$\begin{aligned} \Delta J_k &\leq -Tl(\hat{\mathbf{x}}_k^0, \hat{\mathbf{u}}_k^0) + \sum_{i=0}^{N-2} [Tl(\hat{\mathbf{x}}_{k+i}, \hat{\mathbf{u}}_{k+i}) - Tl(\hat{\mathbf{x}}_{k+i+1}^0, \hat{\mathbf{u}}_{k+i+1}^0)] \\ &\quad + Tl(\hat{\mathbf{x}}_{k+N-1}, \hat{\mathbf{u}}_{k+N-1}) + g(\hat{\mathbf{x}}_{k+N}) - g(\hat{\mathbf{x}}_{k+N-1}) \\ &\quad + g(\hat{\mathbf{x}}_{k+N-1}) - g(\hat{\mathbf{x}}_{k+N}^0) \end{aligned} \quad (30)$$

By virtue of inequality (18) and the Properties 4–7 in Assumption 3, one has

$$l(\hat{\mathbf{x}}_k^0, \hat{\mathbf{u}}_k^0) \geq \tilde{\alpha}_3(\|\hat{\mathbf{x}}_k^0\|) =: \alpha_3(\|\mathbf{x}_k\|) \quad (31)$$

$$l(\hat{\mathbf{x}}_{k+i}, \hat{\mathbf{u}}_{k+i}) - l(\hat{\mathbf{x}}_{k+i+1}^0, \hat{\mathbf{u}}_{k+i+1}^0) \leq L_l e^{iL_f T} [TQ(T) + \zeta T] \quad (32)$$

and

$$g(\hat{\mathbf{x}}_{k+N-1}) - g(\hat{\mathbf{x}}_{k+N}^0) \leq L_g e^{(N-1)L_f T} [TQ(T) + \zeta T] \quad (33)$$

Since $\hat{\mathbf{x}}_{k+N} \in \Xi$, according to Assumption 3 we have

$$g(\hat{\mathbf{x}}_{k+N}) - g(\hat{\mathbf{x}}_{k+N-1}) \leq -Tl(\hat{\mathbf{x}}_{k+N-1}, \hat{\mathbf{u}}_{k+N-1}) \quad (34)$$

Substituting (31), (32), (33) and (34) into (30) yields

$$\Delta J_k \leq T[-\alpha_3(\|\mathbf{x}_k\|) + \tilde{\gamma}(\zeta) + \delta_1] \quad (35)$$

with $\tilde{\gamma}(\zeta) = [TL_l \frac{e^{(N-1)L_f T} - 1}{e^{L_f T} - 1} + L_g e^{(N-1)L_f T}] \zeta$ and $\delta_1 = [TL_l \frac{e^{(N-1)L_f T} - 1}{e^{L_f T} - 1} + L_g e^{(N-1)L_f T}] Q(T)$.

So far, we have shown that we can choose the optimal value function as an ISpS-Lyapunov function for the system (2), thus the exact DTM (2) is ISpS, namely inequality (20) holds.

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