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RESEARCH ARTICLE

A characteristic modeling method of error-free compression for nonlinear systems

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Abstract
The existence of error when compressing nonlinear functions into the coefficients of the characteristic model is known to be a key issue in existing characteristic modeling approaches, which is solved in this work by an error-free compression method. We first define a key concept of relevant states with corresponding compressing methods into their coefficients, where the coefficients are continuous and bounded and the compression is error-free. Then, we give the conditions for decoupling characteristic modeling for MIMO systems, and sequentially, we establish characteristic models for nonlinear systems with minimum phase and relative order two as well as the flexible spacecrafts, realizing the equivalence in the characteristic model theory. Finally, we explicitly explain the reasons for normalization in the characteristic model theory.

Keywords Characteristic modeling · Relevant states · Error-free compression · Flexible spacecraft · Normalization

1 Introduction

The characteristic model theory founded by Academician Wu Hongxin in the 1980s [1–3] has already witnessed many of its successful stories in the aerospace and industry fields, e.g., the reentry lift control of the Shenzhou spacecraft [4–6], the rendezvous and docking control of the Shenzhou spacecraft and Tiangong 1 [7, 8], the skip reentry control of the Chang'e-5 [9], the electrolytic aluminum control [10], just to name a few. The characteristic model theory consists of three ingredients, namely, characteristic modeling, parameter identification and all-coefficient adaptive control. In the characteristic modeling stage, the dynamics of the controlled systems is transformed to establish the characteristic model. Then, in the parameter identification stage, the projection gradient method or the projection least square method are used to identify the bounded coefficients, which is key to the success of the characteristic model. Finally, the control law is designed using the so-called all-coefficient adaptive control approach, examples of which include maintenance-tracking control, golden-section adaptive control, logic integral control, logic differential control, etc. [4].

As can be understood, characteristic modeling is the first step of and key to the characteristic model theory, which has been studied extensively in the recent decades. For linear systems, the problem has been solved by proving that general linear time-invariant systems can be transformed to the second-order linear time-varying difference equations with bounded coefficients under certain conditions [11, 12]. For second-order affine nonlinear systems the second-order characteristic model has also been given by introducing nonlinear time scale [13]. The cyclic demonstration problem is solved by a state-dependent identification projection region and a novel adaptive control method [14, 15]. Also, one assumption on characteristic modeling is that the compressed functions should be zero for zero system state, since otherwise the modeling errors will be infinity under certain conditions, which can be solved by the translation transformation method [16].

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We notice that key to characteristic modeling is the error-free compression of nonlinear functions into the coefficients of a characteristic model, but unfortunately error is always present in all existing methods [4]. On the other hand, the reasons of the so-called “normalization” phenomenon need also be explained, where the bounds of the output coefficients of the characteristic model for different controlled systems, systems different time scales, linear or nonlinear, are all the same. Motived by the above challenges, in the present work, we

– Define a key concept of “relevant states” for nonlinear functions, which ensures the equivalence in the compression process.
– Establish the necessary and sufficient conditions for the first time, under which MIMO systems can be transformed into a decoupled characteristic model.
– Establish the characteristic models of nonlinear systems with minimum phase and relative order two as well as the flexible spacecraft, realizing the equivalence in the characteristic model theory.
– Address the normalization problem in the characteristic model theory.

In what follows, we first formulate the problem of interest in Sect. 2, then present the main results in Sect. 3, and finally concludes the paper in Sect. 4.

2 Problem formulation and preliminaries

We first formulate the considered problems with preliminaries on characteristic modeling with error-free compression.

2.1 Problem formulation

In [19], it was proved that the dynamics of a flexible spacecraft can be transformed into a standard form of input–output linearization with minimum phase and relative order two. With this in mind, we may consider the following affine nonlinear system:

$$\begin{cases} \dot{x}_1 = x_2, \\ \dot{x}_2 = f(x_1, x_2, \eta) + g(x_1, x_2, \eta)u, \\ \dot{\eta} = q(x_1, x_2, \eta), \end{cases} \tag{1}$$

where $x_1 \in \mathbb{R}^n$, $x_2 \in \mathbb{R}^n$ and $\eta \in \mathbb{R}^p$ are the system states, $u \in \mathbb{R}^n$ is the system input, $f \in \mathbb{R}^n$, $g \in \mathbb{R}^{n\times n}$ and $q \in \mathbb{R}^p$ are the smooth differential functions, $f(0,0,0) = 0$ and $q(0,0,0) = 0$.

For the system in (1), we make the following assumption.

Assumption 1 (a) The system in (1) is a minimum-phase system; (b) the derivatives of functions f and q with their arguments are bounded; and (c) g is nonsingular.

Remark 1 Assumption 1 is necessary for the global stability of the characteristic model based adaptive control. In fact, adaptive control can be rewritten as an adaptive PID control law with bounded coefficients [4], and Assumption 1 is one of the necessary conditions for the global stability of PID control in [17].

An advantage of characteristic model theory is that the coefficients of the characteristic model have determined bounds, ensuring transient stability of the closed-loop systems. The bounds are determined by adjusting the sampling period according to how fast the system dynamics can be. This system property can be measured by the eigenvalues for linear time-invariant (LTI) systems, and by the following time scale introduced in [18] for nonlinear systems.

Definition 1 Define the time scale for the system in (1) as follows:

$$T_{\text{scale}} = \min\left\{ \frac{1}{\sqrt{M_f}}, \frac{1}{\sqrt{M_q}}, \frac{1}{\sqrt{M_u}} \right\},$$

where

$$M_f = \max \|f\|, \quad M_q = \max \|q\|, \quad M_u = \max \|gu\|$$

for x_1, x_2, and η in bounded closed sets.

The characteristic model is of the form of linear time-varying difference equations with bounded coefficients, with its particular focus on second-order model [4, 11–16], given as follows:

$$y_C(k+2) = a_1(k)y_C(k+1) + a_2(k)y_C(k) + b(k)u_C(k), \tag{2}$$

where y_C and u_C are the output and input of the characteristic model in (2), with their dimensions being equal to those of the controlled system, and a_1, a_2 and b are matrices with appropriate dimensions. The bounds of the output coefficients are given by

$$a_1(k) = 2I_3 + \mathrm{O}\left(\frac{T}{T_{\text{scale}}}\right), \tag{3}$$

$$a_2(k) = -I_3 + \mathrm{O}\left(\frac{T}{T_{\text{scale}}}\right) + \mathrm{O}\left(\frac{T}{T_{\text{scale}}}\right)^2, \tag{4}$$

and the bound of the input coefficient $b(k)$ is given according to the specific physical properties of the input matrix of the controlled systems. In the above formulas, T_{scale} is the time scale of the controlled systems, and T is the sampling period.

When a_1 and a_2 are diagonal matrices, (2) is said to be the decoupled characteristic model.

2.2 Preliminaries

The following lemma gives an important property of the time scale.

Lemma 1 *Assume that Assumption* 1 *holds. For the following systems,*

$$\begin{cases} \dot{x}_1 = x_2, \\ \dot{x}_2 = f_1 x_1 + f_2 x_2 \end{cases} \tag{5}$$

with time scale T_{scale}, *it is held that*

$$\|f_1\| = \mathrm{O}\left(\frac{1}{T^2_{\text{scale}}}\right), \quad \|f_2\| = \mathrm{O}\left(\frac{1}{T_{\text{scale}}}\right). \tag{6}$$

Similarly, for the following high-order system,

$$\begin{cases} \dot{x}_1 = x_2, \\ \vdots \\ \dot{x}_{n-1} = x_n, \\ \dot{x}_n = f_1 x_1 + f_2 x_2 + \cdots + f_n x_n, \end{cases}$$

it is held that

$$\|f_i\| = \mathrm{O}\left(\frac{1}{T^{n+1-i}_{\text{scale}}}\right).$$

Proof Let the derivative variable of (5) be t, and transform (5) into the system with time scale 1, as follows:

$$\begin{cases} \dfrac{\mathrm{d}x'_1}{\mathrm{d}t'} = x'_2, \\ \dfrac{\mathrm{d}x'_2}{\mathrm{d}t'} = T^2_{\text{scale}} f_1 x'_1 + T_{\text{scale}} f_2 x'_2. \end{cases}$$

By the time-scale transformation, we see that

$$t' = \frac{t}{T_{\text{scale}}},$$

where

$$x'_1 = x_1, \quad x'_2 = T_{\text{scale}} x_2.$$

It follows from Definition 1 that for x_1 and x_2 in bounded closed sets,

$$\max \|T^2_{\text{scale}} f_1 x'_1 + T_{\text{scale}} f_2 x'_2\| = 1,$$

which then means that (6) holds. The other case can be proved similarly.

Remark 2 When degenerated to linear systems, Lemma 1 holds as well. It can be known from linear system theory that f_1 and f_2 are the product and sum of eigenvalues, respectively. Eq. (6) holds from the definition of minimum time constant of linear system theory. Error-free compression of nonlinear functions into the coefficients of state variables is key to characteristic modeling. We define the following relevant states which is useful in the error-free compression.

Definition 2 For a function $h(s_1, s_2, \ldots, s_n)$, if

$$h(0, \ldots, 0, s_{m+1}, \ldots, s_n) = 0, \ m \leqslant n,$$
$$\text{even if} \ \ s_{m+1}, \ldots, s_n \neq 0,$$

then $s_1, s_2, \ldots, s_m$ are said to be a group of relevant states, and $s_1, s_2, \ldots, s_m$ are relevant.

Remark 3 Relevant states exist in general. In fact, all states are relevant for a system with zero equilibrium according to Definition 2. From nonlinear system theory, the nonzero equilibrium can always be transferred to zero.

Remark 4 There may exist multiple groups of relevant states for a system. For example, the following function

$$h(s_1, s_2) = s_1 s_2$$

has three groups of relevant states s_1; s_2; s_1 and s_2. In this present work, we need to find out the group of relevant states with the fewest elements to build the characteristic model.

For a function $h(s_1, s_2, \ldots, s_n)$ with all states being relevant, define for s_1

$$h_1 = \begin{cases} \dfrac{h(s_1, 0, \ldots, 0)}{s_1}, & s_1 \neq 0, \\ \dfrac{\partial h}{\partial s_1}(0, \ldots, 0), & s_1 = 0, \end{cases} \tag{7}$$

and for $i = 2, \ldots, n$,

$$h_i = \begin{cases} \dfrac{h(s_1, \ldots, s_i, 0, \ldots, 0) - h(s_1, \ldots, s_{i-1}, 0, \ldots, 0)}{s_i}, & s_i \neq 0, \\ \dfrac{\partial h}{\partial s_i}(s_1, \ldots, s_{i-1}, 0, \ldots, 0), & s_i = 0. \end{cases} \tag{8}$$

It is easy to see that [17]

$$h(s_1, s_2, \ldots, s_n) = \sum_{i=1}^{n} h_i s_i. \tag{9}$$

For a function $h(s_1, s_2, \ldots, s_n)$ with relevant states being s_i, $i = 1, 2, \ldots, m$, $m < n$, define for s_1

$$h_1 = \begin{cases} \dfrac{h(s_1, 0, \dots, 0, s_{m+1}, \dots, s_n)}{s_1}, \ s_1 \neq 0, \\ \dfrac{\partial h}{\partial s_1}(0, \dots, 0, s_{m+1}, \dots, s_n), \ s_1 = 0, \end{cases} \tag{10}$$

and for $i = 2, \dots, m$,

$$h_i = \begin{cases} \dfrac{h(s_1, \dots, s_i, 0, \dots, 0, s_{m+1}, \dots, s_n) - h(s_1, \dots, s_{i-1}, 0, \dots, 0, s_{m+1}, \dots, s_n)}{s_i}, & s_i \neq 0, \\ \dfrac{\partial h}{\partial s_i}(s_1, \dots, s_{i-1}, 0, \dots, 0, s_{m+1}, \dots, s_n), & s_i = 0. \end{cases} \tag{11}$$

It is then easy to see that

$$h(s_1, s_2, \dots, s_n) = \sum_{i=1}^{m} h_i s_i. \tag{12}$$

The above results are summarized in the following lemma.

Lemma 2 *Consider a function $h(s_1, s_2, \dots, s_n)$ with bounded $\frac{\partial h}{\partial s_i}$, $i = 1, \dots, n$. If and only if the relevant states of h are s_i, $i = 1, \dots, m$, $m \leqslant n$, h can be compressed into the coefficients of the relevant states s_i without error, as shown in (9) where the coefficients are shown in (7) and (8) for $m = n$, and in (12) with the coefficients (10) and (11) for $m < n$. Furthermore, the coefficients h_i, $i = 1, 2, \dots, m$, are continuously bounded.*

3 Main results

This section investigates the characteristic modeling problem with error-free compression for the nonlinear systems (1) and flexible spacecraft, and the normalization phenomena in the characteristic model theory. The characteristic modeling problem for the external dynamics of (1) is first considered.

3.1 Characteristic modeling for second-order affine nonlinear systems

Consider the external dynamics of (1),

$$\begin{cases} \dot{x}_1 = x_2, \\ \dot{x}_2 = f(x_1, x_2) + g(x_1, x_2)u, \\ y = x_1. \end{cases} \tag{13}$$

Let

$$\begin{aligned} &x_1 = [x_{11}, x_{12}, \dots, x_{1n}]^{\mathrm{T}}, \ \ x_2 = [x_{21}, x_{22}, \dots, x_{2n}]^{\mathrm{T}}, \\ &u = [u_1, u_2, \dots, u_n]^{\mathrm{T}}, \ \ f = [f_1, f_2, \dots, f_n]^{\mathrm{T}}, \\ &g = [g_1^{\mathrm{T}}, g_2^{\mathrm{T}}, \dots, g_n^{\mathrm{T}}]^{\mathrm{T}}, \end{aligned} \tag{14}$$

where $x_{1i} \in \mathbb{R}$, $x_{2i} \in \mathbb{R}$, $u_i \in \mathbb{R}$, $f_i \in \mathbb{R}$, $g_i \in \mathbb{R}^{1\times n}$, $i = 1, 2, \dots, n$. By Definition 1, $f(0, 0) = 0$ means there exist relevant states for f_i, $i = 1, 2, \dots, n$, which can be expressed as

$$x_{1k_1}, x_{1k_2}, \dots, x_{1k_i}, x_{2l_1}, x_{2l_2}, \dots, x_{2l_i}, \tag{15}$$

where k_j and l_j belong to the set $\{1, 2, \dots, n\}$, $j = 1, 2, \dots, i$. By Lemma 2, designing the continuous bounded coefficients with respect to the relevant states (15) yields

$$f^i_{1k_1}, f^i_{1k_2}, \dots, f^i_{1k_i}, f^i_{2l_1}, f^i_{2l_2}, \dots, f^i_{2l_i} \tag{16}$$

and

$$f_i(x_1, x_2) = \sum_{j=1}^{i} \left(f^i_{1k_j} x_{1k_j} + f^i_{2l_j} x_{2l_j} \right). \tag{17}$$

From (13) and (17), it is easy to obtain

$$\ddot{x}_{1i} = \sum_{j=1}^{i} \left(f^i_{1k_j} x_{1k_j} + f^i_{2l_j} x_{2l_j} \right) + g_i(x_1, x_2)u. \tag{18}$$

Rewriting (18) in its matrix form and then taking Euclidean discretization yields

$$x_1(k + 2) = a_1(k)x_1(k + 1) + a_2(k)x_1(k) + b(k)u(k), \tag{19}$$

where $a_1(k) \in \mathbb{R}^{n\times n}$ and $a_2(k) \in \mathbb{R}^{n\times n}$ satisfy (3) and (4) by Lemma 1, and

$$b(k) = T^2 g(x_1, x_2). \tag{20}$$

It is easy to see that by further taking the output of the characteristic model as $y_C = x_1$, (19) has the form of the characteristic model in (2), i.e., we have established the characteristic model (2) for (13).

We proceed to consider the problem of decoupling characteristic modeling, which diminishes the number of coefficient identification, hence simplifying the control design.

Lemma 2 implies that if f_i, $i = 1, 2, \dots, n$, satisfy the following conditions:

$$
\begin{aligned}
& f_i(x_{1k_1}, \dots, x_{1i-1}, 0, x_{1i+1}, \dots, x_{1k_i}, \\
& x_{2l_1}, \dots, x_{2i-1}, 0, x_{2i+1}, \dots, x_{2l_i}) = 0, \\
& \text{even if } \; x_{1j} \neq 0, x_{2j} \neq 0, \;\; j \neq i, \;\; i,j = 1, 2, \dots, n,
\end{aligned} \tag{21}
$$

then by designing the continuously bounded coefficients f_{1i}^i and f_{2i}^i, we obtain

$$
f_i = f_{1i}^i x_{1i} + f_{2i}^i x_{2i}. \tag{22}
$$

By (13), we have

$$
\ddot{x}_{1i} = f_{1i}^i x_{1i} + f_{2i}^i x_{2i} + g_i(x_1, x_2)u. \tag{23}
$$

Rewriting (23) in its matrix form and then taking Euclidean discretization yields

$$
x_1(k+2) = a_1(k)x_1(k+1) + a_2(k)x_1(k) + b(k)u(k). \tag{24}
$$

Apparently, $a_1(k)$ and $a_2(k)$ are diagonal, which means that (24) is decoupled. It is easy to see that by further taking the output and input of the characteristic model as $y_C = x_1$ and $u_C = u$, (23) has the form of the characteristic model (2); that is, we have established the decoupled characteristic model (2) for (13), where $a_1(k) \in \mathbb{R}^{n\times n}$ and $a_2(k) \in \mathbb{R}^{n\times n}$ are diagonal matrices and satisfy (3) and (4), and $b(k)$ is with the form of (20). Eq. (21) is necessary and sufficient for establishing the decoupled characteristic model, which is said to be the decoupling condition of characteristic model.

We summarize the above derivations into the following two theorems.

Theorem 1 *If the nonlinear system in* (13) *satisfies Assumption* 1, *then there exist relevant states for the nonlinear function f in* (13), *and f can be compressed into the coefficients of the relevant states without error with continuously bounded coefficients* (16), *as shown in* (17). *Furthermore,* (13) *can be transformed into the second-order characteristic model* (2) *with the coefficients satisfying* (3), (4), *and* (20).

Theorem 2 *If the nonlinear system in* (13) *satisfies Assumption* 1 *and the decoupling condition* (21), *then for* $i = 1, 2, \dots, n$, f_i *only has two relevant states,* x_{1i} *and* x_{2i}, *and f can be compressed into the coefficients of the relevant states without error with continuously bounded coefficients, as shown in* (22). *Furthermore,* (13) *can be transformed into the second-order decoupled characteristic model* (2) *with the coefficients satisfying* (3), (4), *and* (20), *and* a_1 *and* a_2 *diagonal.*

Remark 5 The decoupling conditions (21) for establishing the decoupled characteristic model are given for the first time in this work.

Remark 6 Equations (17) and (22) show that the compression is error-free, ensuring the equivalence in the characteristic model theory.

3.2 Characteristic modeling for the nonlinear systems in (1)

This section gives a characteristic modeling method with error-free compression for the higher-order nonlinear systems in (1), realizing the equivalence in the characteristic model theory.

Similar to (14), let

$$
q = [q_1, q_2, \dots, q_p]^{\mathrm{T}}, \quad \eta = [\eta_1, \eta_2, \dots, \eta_p]^{\mathrm{T}},
$$

where $q_i \in \mathbb{R}$, $\eta_i \in \mathbb{R}$, $i = 1, 2, \dots, p$. In the following, we establish the characteristic model of (1). First, compress the nonlinear function f and q of (1) into the coefficients of the relevant states. Similar to the deduction procedure in Section 3.1, for f, there exist relevant states,

$$
x_{1k_1}, x_{1k_2}, \dots, x_{1k_i}, x_{2l_1}, x_{2l_2}, \dots, x_{2l_i}, \eta_{r_1}, \eta_{r_2}, \dots, \eta_{r_i}.
$$

Designing the continuous bounded coefficients as follows:

$$
f_{1k_1}^i, f_{1k_2}^i, \dots, f_{1k_i}^i, f_{2l_1}^i, f_{2l_2}^i, \dots, f_{2l_i}^i, f_{r_1}^i, f_{r_2}^i, \dots, f_{r_i}^i
$$

yields

$$
f_i(x_1, x_2, \eta) = \sum_{j=1}^{i} \left(f_{1k_j}^i x_{1k_j} + f_{2l_j}^i x_{2l_j} + f_{r_j}^i \eta_{r_j} \right). \tag{25}
$$

For q, there exist relevant states:

$$
x_{1s_1}, x_{1s_2}, \dots, x_{1s_i}, x_{2v_1}, x_{2v_2}, \dots, x_{2v_i}, \eta_{w_1}, \eta_{w_2}, \dots, \eta_{w_i}.
$$

Future designing the continuous bounded coefficients,

$$
q_{1s_1}^i, q_{1s_2}^i, \dots, q_{1s_i}^i, q_{2v_1}^i, q_{2v_2}^i, \dots, q_{2v_i}^i, q_{w_1}^i, q_{w_2}^i, \dots, q_{w_i}^i,
$$

and hence

$$
q_i(x_1, x_2, \eta) = \sum_{j=1}^{i} \left(q_{1s_j}^i x_{1s_j} + q_{2v_j}^i x_{2v_j} + q_{w_j}^i \eta_{w_j} \right). \tag{26}
$$

Therefore, (1) can be rewritten in matrix form by substituting (25) and (26) into it,

$$
\begin{cases}
\dot{x}_1 = x_2, \\
\dot{x}_2 = F_1 x_1 + F_2 x_2 + F_\eta \eta + gu, \\
\dot{\eta} = Q_1 x_1 + Q_2 x_2 + Q_\eta \eta,
\end{cases} \tag{27}
$$

where F_1, F_2, F_η, Q_1, Q_2, and Q_η are matrices with appropriate dimensions. We can see that the nonlinear functions f and q in (1) are compressed into the coefficients of the relevant states without error by comparing (1) and (27).

We further deal with the internal states η as follows. By matrix theory, we can find the general solution of η from the second equation of (27),

$$\eta = f_{\eta 3}\ddot{x}_1 + f_{\eta 2}\dot{x}_1 + f_{\eta 1}x_1 + f_u u, \tag{28}$$

where $f_{\eta 1}$, $f_{\eta 2}$, $f_{\eta 3}$, and f_u are the matrices with appropriate dimensions. Differentiating the second equation of (27) yields

$$\dddot{x}_2 = F_2\ddot{x}_1 + (F_1 + \dot{F}_2)\dot{x}_1 + \dot{F}_1 x_1 + \dot{F}_\eta \eta + F_\eta \dot{\eta} + \dot{g}u + g\dot{u}. \tag{29}$$

Substituting the third equation of (27) into (29) cancels $\dot{\eta}$, and then substituting (28) into the resultant equation cancels η. This results in the third-order differential equation of x_1 with the cancellation of the internal states η and $\dot{\eta}$,

$$\dddot{x}_1 = A_3\ddot{x}_1 + A_2\dot{x}_1 + A_1 x_1 + g\dot{u} + (\dot{g} + (\dot{F}_\eta + F_\eta Q_\eta)f_u)u, \tag{30}$$

where

$$\begin{aligned} A_3 &= F_2 + \dot{F}_2 + (\dot{F}_\eta + F_\eta Q_\eta)f_{\eta 3},\\ A_2 &= F_1 + \dot{F}_2 + F_\eta Q_2 + (\dot{F}_\eta + F_\eta Q_\eta)f_{\eta 2},\\ A_1 &= \dot{F}_1 + F_\eta Q_1 + (\dot{F}_\eta + F_\eta Q_\eta)f_{\eta 1}. \end{aligned}$$

It follows from Lemma 1 that

$$\begin{aligned} \|A_3\| &= \mathrm{O}\left(\frac{1}{T_{\text{scale}}}\right),\\ \|A_2\| &= \mathrm{O}\left(\frac{1}{T^2_{\text{scale}}}\right),\\ \|A_1\| &= \mathrm{O}\left(\frac{1}{T^3_{\text{scale}}}\right). \end{aligned}$$

Taking Euler discretization to (30), the third-order characteristic model can be established by further taking $y_C = x_1$ and $u_C = u$,

$$\begin{aligned} y_C(k+3) = {} & a_1(k)y_C(k+2) + a_2(k)y_C(k+1)\\ & + a_3(k)y_C(k) + b_1(k)u_C(k+1) + b_0(k)u_C(k), \end{aligned} \tag{31}$$

where

$$\begin{cases} a_1(k) = 3I_3 + \mathrm{O}\left(\dfrac{T}{T_{\text{scale}}}\right),\\ a_2(k) = -3I_3 + \mathrm{O}\left(\dfrac{T}{T_{\text{scale}}}\right) + \mathrm{O}\left(\dfrac{T}{T_{\text{scale}}}\right)^2,\\ a_3(k) = I_3 + \mathrm{O}\left(\dfrac{T}{T_{\text{scale}}}\right) + \mathrm{O}\left(\dfrac{T}{T_{\text{scale}}}\right)^2 + \mathrm{O}\left(\dfrac{T}{T_{\text{scale}}}\right)^3, \end{cases} \tag{32}$$

$$\begin{cases} b_1(k) = T^2 g(x_1, x_2, u),\\ b_0(k) = T^3\left(\dot{g} + (\dot{F}_\eta + F_\eta Q_\eta)f_u\right) - T^2 g(x_1, x_2, u). \end{cases} \tag{33}$$

In characteristic model theory, the second-order one is of special significance since the intelligent adaptive control methods based on it have derived successful and widely applications [4]. Therefore, we further establish the second-order characteristic model for (1) by introducing the online estimation methods. It follows from the first and second equations in (27) that

$$\ddot{x}_1 = F_1 x_1 + F_2\dot{x}_1 + F_\eta \eta + gu. \tag{34}$$

By designing online estimators, for example, the extended state observer (ESO), to estimate the internal states $F_\eta \eta$, and denoting the estimated states as $\bar{\eta}$, (34) can be represented as follows:

$$\ddot{x}_1 = F_1 x_1 + F_2\dot{x}_1 + \bar{u}, \tag{35}$$

where

$$\bar{u} = gu + \bar{\eta}. \tag{36}$$

Furthermore, by taking Euler discretization to (35), the second-order characteristic model (2) can be established where $y_C = x_1$, $u_C = \bar{u}$, and

$$b(k) = T^2. \tag{37}$$

We summarize the above deductions into the following theorem.

Theorem 3 *If the nonlinear systems in* (1) *satisfy Assumption* 1, *then* (1) *can be transformed into the third-order characteristic model in* (31) *with the coefficients satisfying* (32) *and* (33); *furthermore, if the internal states are estimated and the intermediate control* $\bar{u}$ *is designed as in* (36), *then* (1) *can be transformed into the second-order characteristic model* (2) *with the input* $u_C = \bar{u}$, *and the coefficients satisfying* (3), (4), *and* (37).

3.3 Characteristic modeling for flexible spacecraft

This section proposes the characteristic modeling method with error-free compression for flexible spacecraft, realizing the equivalence in the characteristic model theory.

Consider the following flexible spacecraft attitude dynamics (1-3-2 Euler rotation sequence),

$$\begin{cases} \dot{x}_1 = C(x_1)w_s,\\ I_s\dot{w}_s + \tilde{w}_s I_s w_s + F_{sL}\ddot{\eta}_L + F_{sR}\ddot{\eta}_R = T_s,\\ \ddot{\eta}_L + 2\xi_L w_L \dot{\eta}_L + w_L^2 \eta_L + F_{sL}^{\mathrm{T}}\dot{w}_s = 0,\\ \ddot{\eta}_R + 2\xi_R w_R \dot{\eta}_R + w_R^2 \eta_R + F_{sR}^{\mathrm{T}}\dot{w}_s = 0, \end{cases} \tag{38}$$

where

$x_1 = [\phi \;\; \theta \;\; \psi]^{\mathrm{T}}$;

$w_s = \begin{bmatrix} w_x & w_y & w_z \end{bmatrix}^{\mathrm{T}}$;

ϕ, θ, ψ are the roll, the pitch, and the yaw attitudes; w_x, w_y, w_z are the roll, the pitch and the yaw angular velocities, respectively;

$$C(x_1) = \begin{bmatrix} \dfrac{\cos\theta}{\cos\psi} & 0 & \dfrac{\sin\theta}{\cos\psi} \\ \tan\psi\cos\theta & 1 & \tan\psi\sin\theta \\ -\sin\theta & 0 & \cos\theta \end{bmatrix}$$

is the transformation matrix;

$$\tilde{w}_s = \begin{bmatrix} 0 & -w_z & w_y \\ w_z & 0 & -w_x \\ -w_y & w_x & 0 \end{bmatrix}$$

represents the cross-product operator matrix; I_s is the inertia matrix of spacecraft, T_s is the external torque vector acting on the spacecraft; and $\eta_L \in \mathbb{R}^l$ and $\eta_R \in \mathbb{R}^l$ are the mode coordinate matrices, $\xi_L > 0$ and $\xi_R > 0$ are the mode damping coefficients, $w_L > 0$ and $w_R > 0$ are the mode frequencies, $F_{sL} \in \mathbb{R}^{3\times l}$ and $F_{sR} \in \mathbb{R}^{3\times l}$ are the coupled matrices between flexible and rigid bodies with the subscript L and R being the left and right solar array, respectively.

Here, we consider the case where $0 \leqslant \psi < 90°$ (for $\psi \geqslant 90°$, the spacecraft model in terms of quaternion parameterization is needed). Through simple computation, $C(x_1)$ is nonsingular for $0 \leqslant \psi < 90°$. Let the main body inertial matrix be

$$R_s = I_s - FF^{\mathrm{T}},$$

where

$$F = \begin{bmatrix} F_{sL}, F_{sR} \end{bmatrix}.$$

Assumption 2 R_s is nonsingular.

The following properties are given in [19].

Lemma 3 *The relative order of the flexible spacecraft dynamics* (38) *is* (2, 2, 2), *and its zero dynamics are exponentially stable.*

Using differential homeomorphic transformation, (38) is transformed into the following input-output linearization form,

$$\begin{cases} \dot{z}_1 = z_2, \\ \dot{z}_2 = a_2(z_1, z_2)z_2 + a(z)\eta + b(z_1)T_s, \\ \dot{\eta} = A_\eta \eta + B_\eta z_2, \end{cases} \tag{39}$$

where

$$\begin{aligned}
&z_1 = x_1, \\
&z_2 = C(x_1)w_s, \\
&\eta = \begin{bmatrix} \eta_L^{\mathrm{T}}, w_s^{\mathrm{T}}F_{sL} + \dot{\eta}_L^{\mathrm{T}}, \eta_R^{\mathrm{T}}, w_s^{\mathrm{T}}F_{sR} + \dot{\eta}_R^{\mathrm{T}} \end{bmatrix}^{\mathrm{T}}, \\
&a_1(z_1, z_2) = \frac{\partial C}{\partial z_1}((C(z_1)^{-1}z_2) \otimes I_{3\times 3}) \\
&\qquad -b(z_1)(2\xi_L w_L F_{sL}F_{sL}^{\mathrm{T}} + 2\xi_R w_R F_{sR}F_{sR}^{\mathrm{T}})C(z_1)^{-1}, \\
&a_2(z_1, z_2) = a_1(z_1, z_2) - b(z_1)\tilde{w}_s I_s C(z_1)^{-1}, \\
&b(z_1) = C(z_1)R_s^{-1}, \\
&a(z) = C(z_1)R_s^{-1}\begin{bmatrix} w_L^2 F_{sL}, 2\xi_L w_L F_{sL}, w_R^2 F_{sR}, 2\xi_R w_R F_{sR} \end{bmatrix}, \\
&A_\eta = \begin{bmatrix} 0 & I_{l\times l} & 0 & 0 \\ -w_L^2 I_{l\times l} & -2\xi_L w_L I_{l\times l} & 0 & 0 \\ 0 & 0 & 0 & I_{l\times l} \\ 0 & 0 & -w_R^2 I_{l\times l} & -2\xi_R w_R I_{l\times l} \end{bmatrix}, \\
&B_\eta = \begin{bmatrix} -F_{sL}, 2\xi_L w_L F_{sL}, -F_{sR}, 2\xi_R w_R F_{sR} \end{bmatrix}^{\mathrm{T}} C(z_1)^{-1}.
\end{aligned}$$

By comparison, we can see that (39) is already in the form of (27), meaning that it is not necessary to compress the nonlinear functions for the flexible spacecraft (38), which consequently means that Assumption 1 is unnecessary. From Theorem 3, we can obtain the characteristic modeling results directly.

Proposition 1 *If the flexible spacecraft in* (38) *satisfies Assumption* 2, *then* (38) *can be transformed into the third-order characteristic model in* (31) *with the coefficients satisfying* (32) *and* (33); *furthermore, if the internal states* $a(z)\eta$ *are estimated as* $\bar{\eta}$ *and the intermediate control* $\overline{T}_s$ *is designed as follows*:

$$\overline{T}_s = b(z_1)T_s + \bar{\eta},$$

then (38) *can be transformed into the second-order characteristic model* (2) *with the input* $u_C = \overline{T}_s$, *and the coefficients satisfying* (3), (4), *and* (37).

3.4 Normalization essence

This section provides insights on "normalization" in the characteristic model theory through both linear and nonlinear systems. "normalization" means that the bounds of the output coefficients of the characteristic model for different controlled systems, systems different time scales, linear or nonlinear, are all the same.

3.4.1 LTI systems

Consider the following controlled systems with different poles:

$$G(s) = \sum_{i=1}^{n} \frac{k_i}{s - \lambda_i}, \tag{40}$$

where $\lambda_i < 0$ and k_i are real numbers for $i = 1, 2, \ldots, n$. It follows from computer control theory that the pulse transfer function for (40) is

$$H(z) = \frac{b_0 + b_1 z^{-1} + \cdots + b_n z^{-n}}{1 + a_1 z^{-1} + \cdots + a_n z^{-n}},$$

where

$$\begin{cases} a_1 = -\sum_{i=1}^{n} \exp(\lambda_i T), \\ \vdots \\ a_n = (-1)^n \exp(\sum_{i=1}^{n} \lambda_i T), \end{cases} \tag{41}$$

$$\begin{cases} b_0 = G(0) \sum_{i=1}^{n} \left(1 - \exp(\lambda_i T)\right), \\ \vdots \\ b_n = G(0) \sum_{i=1}^{n} \Big((-1)^n \exp(\sum_{j=1}^{n} \lambda_j T) \\ \qquad +(-1)^{n-1} \exp(\lambda_i T) \exp(\sum_{j=1, j\neq i}^{n} \lambda_j T)\Big) \end{cases} \tag{42}$$

and $G(0)$ is the static gain.

In the characteristic model theory, the static gain can be transformed to 1 through the input-output transformation [20, 21]. Thus, without loss of generality, we may assume $G(0) = 1$. Inspection of (41) and (42) shows that for different systems, only the eigenvalue λ_i and the sampling period T are different, but they all appear in the form of multiplication with same orders. Defining the minimum time constant [4],

$$T_{\text{scale}} = \frac{1}{\max_i(-\lambda_i)}$$

results in

$$|\lambda_i| T \leqslant \frac{T}{T_{\text{scale}}}, \quad i = 1, 2, \ldots, n. \tag{43}$$

In the characteristic model theory, we generally choose the sampling period T according to T_{scale} [4],

$$\frac{T}{T_{\text{scale}}} \in \left[\frac{1}{15}, \frac{1}{3}\right]. \tag{44}$$

Using (43) and (44), we can obtain the same bounds for all coefficients in (41) and (42) for linear different systems (40), which implies the realization of the normalization for the linear system in (40).

3.4.2 Nonlinear systems

The reasons for the normalization for the nonlinear system in (1) can be seen from (3), (4), and (32). By choosing the sampling period according to (44), the bounds of the coefficients of the characteristic model in (2) and (31) for different controlled systems are equal, which implies the realization of normalization for the nonlinear systems in (1).

In summary, the essence of the characteristic model theory is to choose the sampling period according to the change pace of controlled systems, and further taking advantage of the structural features of the characteristic model to realize the normalization. For both linear and nonlinear systems, the output coefficient bounds of their same-order characteristic model are the same.

4 Conclusion

The characteristic modeling problem with error-free compression for nonlinear systems is investigated. A key concept of the relevant states is defined with its corresponding compression method, where the coefficients are continuous and bounded and the compression is error-free. The conditions given for decoupling characteristic modeling for MIMO systems provide bases, based on which the establishment of the characteristic models for the nonlinear systems with minimum phase and relative order two and the flexible spacecraft realizes the equivalence in the characteristic model theory. Lastly, reasons for normalization in the characteristic model theory are given. The work contributes fundamentally to the characteristic model theory.

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