A Novel Self-Triggered MPC Scheme for Constrained Input-Affine Nonlinear Systems

Pengfei Li, Yu Kang, *Senior Member, IEEE*, Yun-Bo Zhao, *Senior Member, IEEE*, and Tao Wang

***Abstract*—This brief develops a novel self-triggered model predictive control algorithm based on time delay estimation for input-affine nonlinear systems. At each triggering instant, the algorithm determines simultaneously the predictive control sequence to feedforward compensate for the disturbance and the next triggering instant. As a result, the unnecessary samplings and transmissions are suppressed, and the frequency of solving the optimal control problem is reduced. The feasibility as well as the associated stability are verified, with a numerical example illustrating the effectiveness of the proposed scheme.**

***Index Terms*—Self-triggered control, nonlinear system, predictive control, time delay estimation.**

I. Introduction

MODEL predictive control (MPC), an effective technique for optimizing the control performance subject to constraints and uncertainties, has been extensively studied [1]. Its implementation requires to solve, at every sampling instant, an optimal control problem (OCP) often with high computational complexity. To deal with the computing challenge, the networked configuration can be useful where the computing procedures are completed by the remote controller (high-performance computer). In such a system configuration, the controller connects to the sensor and actuator via communication networks, resulting in the so-called networked control system [2]. Despite its potential advantage of the networked configuration, the limited communication resource (e.g., bandwidth and energy of network nodes) requires one to design an efficient transmission strategy and an effective MPC scheme at the same time.

Event/self-triggered MPC schemes have been developed, to guarantee the satisfactory control performance, and to reduce the communication burden and energy consumption of the sensor by avoiding continuous transmissions. In an event-triggered scheme, the triggering instant, i.e., the time when a transmission is allowed, is determined only if a prescribed triggering condition is violated, marking the importance of the design of the triggering condition. One design approach is based on the recursive feasibility, where the resultant triggering condition are usually involving the error between the predicted state and actual one, see [3]–[6]. An alternative approach is to ensure the stability by keeping the optimal MPC value function decreasing between two consecutive time steps [7] or triggering instants [8]. Different from the event-triggered scheme where the condition should be checked at every time instant, the self-triggered scheme is developed to bypass such a process by precomputing the next triggering instant based on the predictive control sequence. Typical examples can be found in [9]–[13] and the references therein, where the next triggering instant is determined by predicting an instant when the conditions related to feasibility and stability are firstly violated. Recently, the triggering and control co-design methods have also been proposed by taking the communication effect into consideration [14].

One may notice that the self-triggered scheme is in general conservative because the adopted parameters, e.g., the bound of disturbance and the related local Lipschitz constants, are conservative in determining the triggering instant. In particular, using only the upper bound of the disturbance brings conservativeness in specifying the error between the predicted state and actual one as the worst-case disturbance would not occur frequently. In addition, the disturbance is slowly time-varying in some practical systems. For example, the disturbance of the wheeled mobile robot is mainly induced from the road ride, and may slightly change if the road condition is quite smooth. For such cases, the disturbance can be predicted with acceptable precision by some disturbance rejection mechanisms in estimating the future states, lower estimation error, or less conservativeness, can be obtained [15], [16]. This motivates us to design a self-triggered scheme by incorporating disturbance estimation and rejection techniques.

In this brief, a new self-triggered MPC scheme is developed by exploiting two upper bounds, namely, the upper bounds of the disturbance and the disturbance variation. The disturbance is estimated by a designed time-delay estimator (TDE), which plays a similar role as the disturbance observer that may not work for discrete-time systems. Compared with the aforementioned works, the estimation error is reduced by compensating for the estimated disturbance, and then the triggering interval is enlarged as a result. Finally, the recursive feasibility as well as the stability of the proposed MPC scheme are verified.

Notations: For a vector x, x^T is its transpose, and $\|x\|$ is its Euclidean norm. For a matrix Φ, $\Phi > 0$ implies that Φ is a positive definite matrix. We use $\bar{\lambda}(\Phi)$ to represent the

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maximum eigenvalue of Φ. Note that $\|x\|_\Phi \triangleq \sqrt{x^T \Phi x}$ is the weighted norm with $\Phi > 0$. Given two nonempty $\mathbb{Y}$ and $\mathbb{Z}$, the definition of the Pontryagin difference set is $\mathbb{Y} \ominus \mathbb{Z} \triangleq \{y : y + z \in \mathbb{Y}, \forall z \in \mathbb{Z}\}$. The floor function and the ceiling function are denoted by $\lfloor . \rfloor$ and $\lceil . \rceil$, respectively.

II. Problem Formulation

In this brief, the following input-affine discrete-time nonlinear system is considered

$$x_{k+1} = f(x_k) + Bu_k + w_k, \ \ k \geq 0 \tag{1}$$

where $x_k \in \mathbb{R}^n$ is the plant state, $u_k \in \mathbb{R}^m$ is the control variable, $w_k \in \mathbb{R}^n$ is the external disturbances, and $B \in \mathbb{R}^{n\times m}$ is the input matrix. Additionally, system (1) and the disturbance w_k satisfy the following assumption.

Assumption 1: 1) (constraints) The state, control input and disturbance are restricted to fulfill the following constraints

$$x_k \in \mathbb{X}, \ \ u_k \in \mathbb{U}, \ \ w_k \in \mathbb{W}, \ \ k \geq 0$$

where $\mathbb{X} \subseteq \mathbb{R}^n$, $\mathbb{U} \subseteq \mathbb{R}^m$ and $\mathbb{W} = \{w \in \mathbb{R}^n : \|w\| \leq \eta\}$ are all compact sets containing the origin as an interior point.

2) (Lipschitz continuous) The function $f(x)$ is local Lipschitz continuous with constant L_Φ relying on the weighted matrix Φ, i.e., $\|f(x) - f(y)\|_\Phi \leq L_\Phi \|x - y\|_\Phi$, $\forall x, y \in \mathbb{X}$.

3) (bound of disturbance variation) The variation of the disturbance is bounded from above, i.e., the inequality $\|w_{k+1} - w_k\| \leq \delta$ holds with constant $\delta > 0$ for all $k \geq 0$.

4) (matched disturbance) For any disturbance vector $w \in \mathbb{W}$, there exists a vector $v \in \mathbb{U}$ such that $Bv = -w$.

Remark 1: The properties 1) and 2) in the above assumption are fairly standard and can be found, e.g., [3], [11]. Property 3) indicates that the disturbance does not vary drastically between two consecutive time steps, and the upper bound δ can be estimated in actual experiments. The last property is vital to the feedforward compensation for the disturbances and can be found, e.g., [17], [18] for Euler-Lagrange systems.

For the ease of expository, we introduce the nominal system of (1) as follows, by neglecting the disturbance,

$$x_{k+1} = f(x_k) + Bu_k \tag{2}$$

The implementation of periodic MPC needs to solve an OCP at each time step [19], consuming a great amount of computing resources. In this brief we adopt a self-triggered dual-mode MPC scheme, whose computing time is determined according to an self-triggered condition, and whose control strategy is switched from solving an OCP to using a feedback control law when the state is steered into the terminal constraint set.

Let k_j, $j \in \mathbb{N}_0$ be the triggering instant when the MPC is solved. The next triggering instant k_{j+1} is determined in a self-triggered manner. To be specific,

$$k_{j+1} = k_j + m(k_j) \tag{3}$$

where $m(.)$ is the inter-execution time or the triggering interval. One may note that larger triggering interval in general implies lower frequency of solving the MPC. With this fact, our aim is to design, by exploiting more information on the disturbance, a self-triggered MPC scheme that has larger triggering intervals, while maintaining feasibility and stability.

III. TDE-Based Self-Triggered MPC

A. Time Delay Estimation

The third property in Assumption 1 means that the variation of the disturbance is not large, and thus it makes sense to take the previous value as its estimation. More specifically, suppose that the two successive states x_{k_j-1}, x_{k_j}, and the last control input u_{k_j-1} are available to the controller at triggering time k_j, then the estimated disturbance $\hat{w}_{k_j}$, i.e., the TDE of the disturbance, is given by

$$\hat{w}_{k_j} \triangleq w_{k_j-1} = x_{k_j} - f(x_{k_j-1}) - Bu_{k_j-1}. \tag{4}$$

B. TDE-Based MPC

At every triggering instant k_j, denote the N-step control sequence calculated based on the nominal system (2) by $\mathbf{v}_{k_j} = \{v_{k_j|k_j}, \ldots, v_{k_j+N-1|k_j}\}$. To compensate for the estimated disturbance (4), the actual control sequence $\mathbf{u}_{k_j}$ can then be designed as

$$u_{k_j+i|k_j} = \begin{cases} v_{k_j+i|k_j} + \nu_{k_j} & i = 0, \ldots, N_c - 1 \\ v_{k_j+i|k_j} & i = N_c, \ldots, N-1 \end{cases} \tag{5}$$

where ν_{k_j} satisfies $B\nu_{k_j} = -\hat{w}_{k_j}$, and N_c is the step length of feedforward compensation designed as $N_c = \min\{\lfloor \frac{\eta}{\delta} \rfloor, N\}$.

The actual control sequence $\mathbf{u}_{k_j}$ consists of $\mathbf{v}_{k_j}$ and ν_{k_j}, where the role of $\mathbf{v}_{k_j}$ is to stabilize the system and ν_{k_j} is to compensate for the disturbance. N_c is essential to the effect of feedforward compensation, which should be neither too large nor too small. Indeed, Lemma 1 suggests that our design of N_c minimizes the state estimation errors over the horizon N.

Since the disturbance is bounded, we can define a compact set $\mathcal{V} \subseteq \mathbb{U}$. For any $\hat{w} \in \mathbb{W}$, we can find a $\nu \in \mathcal{V}$ satisfying $B\nu = -\hat{w}$. Therefore, if $v_{k_j+i|k_j} \in \mathbb{U} \ominus \mathcal{V}$, $i = 0, \ldots, N-1$, the control sequence (5) satisfies the control constraint.

In the sequel, we formulate the OCP to be solved to obtain the control sequence $\mathbf{v}_{k_j}$ at each triggering instant. The robust constraint tightening scheme is proposed in order that the constraint satisfaction and recursive feasibility can be ensured. Firstly, we discuss the upper bound of the state error.

Lemma 1: Given the N-step control sequences $\mathbf{v}_{k_j}$ and $\mathbf{u}_{k_j}$, and let $\hat{x}_{k_j+i|k_j}$ be the i-step state prediction based on the nominal system (2) with initial state x_{k_j} and control sequence $\mathbf{v}_{k_j}$. The error $e_{k_j+i} \triangleq x_{k_j+i} - \hat{x}_{k_j+i|k_j}$ is then bounded by

$$\|e_{k_j+i}\|_P \leq \sum_{s=0}^{i-1} L_P^s \sqrt{\bar{\lambda}(P)} \min\{(i-s)\delta, \eta\} \tag{6}$$

where L_P is defined in Assumption 1 with weighted matrix P.

Proof: Considering the Lipschitz constant in Assumption 1, the above inequality can be verified.

a) For the case that $N_c = 0$, the N-step control sequence $\mathbf{u}_{k_j}$ is directly obtained by the MPC scheme, then the error bound can be similarly obtained as in Lemma 1 in [7]. That is, $\|e_{k_j+i}\|_P \leq \sum_{s=0}^{i-1} \sqrt{\bar{\lambda}(P)} L_P^s \eta$.

b) For the case that $N_c = N$, we then obtain $\|e_{k_j+i}\|_P \leq \|f(x_{k_j+i-1}) + Bu_{k_j+i-1|k_j} + w_{k_j+i-1} - f(\hat{x}_{k_j+i-1|k_j}) - Bv_{k_j+i-1|k_j}\|_P \leq L_P \|x_{k_j+i-1} - \hat{x}_{k_j+i-1|k_j}\|_P + i\sqrt{\bar{\lambda}(P)}\delta \leq \cdots \leq (L_P^{i-1} + 2L_P^{i-2} + \cdots + (i-1)L_P + i)\sqrt{\bar{\lambda}(P)}\delta = \sum_{s=0}^{i-1} L_P^s \sqrt{\bar{\lambda}(P)}(i-s)\delta$, where $\|w_k - \hat{w}_{k_j}\|_P = \|w_k - w_{k_j-1}\|_P \leq \sqrt{\bar{\lambda}(P)}(k - k_j + 1)\delta$ holds for all $k \geq k_j$.

c) For the case that $1 \leq N_c < N$, recalling the above two cases, when $N_c < i \leq N$, we directly obtain $\|e_{k_j+i}\|_P \leq \sum_{s=0}^{N_c-1} L_P^{i-N_c+s}(N_c - s)\sqrt{\bar{\lambda}(P)}\delta + \sum_{s=0}^{i-N_c-1} L_P^s \sqrt{\bar{\lambda}(P)}\eta$. When $i \leq N_c$, we obtain the same bound as case b).

Noting that $N_c\delta = \lfloor \eta/\delta \rfloor \delta \leq \eta$ and $(N_c+1)\delta \geq \lceil \eta/\delta \rceil \delta \geq \eta$, the above three cases can be summarized as (6). ■

Remark 2: From (6), one may note that if the disturbance changes fast, i.e., δ is larger than η, then we have $N_c = 0$, which implies that the feedforward compensation for the TDE of disturbance does not make sense. In contrast, for the slowly-changing disturbance with $N_c \geq 1$, the usage of feedforward compensation reduces the error bound.

According to the above lemma, we define the following set

$$\mathbb{B}(i) = \{e : \|e\|_P \leq \sum_{s=0}^{i-1} L_P^s \sqrt{\bar{\lambda}(P)} \min\{(i-s)\delta, \eta\}\} \tag{7}$$

and can be observed that $e_{k_j+i} \in \mathbb{B}(i)$, $\forall j \geq 0$.

With the above preliminaries, the OCP at triggering time k_j with initial condition x_{k_j} is formulated as follows:

$$\begin{aligned} \min_{\mathbf{v}_{k_j}} \quad & V_N(x_{k_j}, \mathbf{v}_{k_j}) \\ \text{s.t.} \quad & \hat{x}_{k_j+i+1|k_j} = f(\hat{x}_{k_j+i|k_j}) + Bv_{k_j+i|k_j} \\ & \hat{x}_{k_j+i|k_j} \in \mathbb{X} \ominus \mathbb{B}(i), \\ & \hat{x}_{k_j+N|k_j} \in \mathbb{X}_f \\ & v_{k_j+i|k_j} \in \mathbb{U} \ominus \mathcal{V} \end{aligned} \tag{8}$$

where $i = 0, \ldots, N-1$ with N being the prediction horizon, $\hat{x}_{k_j|k_j} = x_{k_j}$, the terminal set is $\mathbb{X}_f \triangleq \{x | \|x\|_P \leq \varepsilon_f\}$, and $V_N(x_{k_j}, \mathbf{v}_{k_j})$ is the cost function defined as

$$V_N(x_{k_j}, \mathbf{v}_{k_j}) = \sum_{i=0}^{N-1} l(\hat{x}_{k_j+i|k_j}, v_{k_j+i|k_j}) + F(\hat{x}_{k_j+N|k_j})$$

with $l(x, v) = \|x\|_Q^2 + \|v\|_R^2$ and $F(x) = \|x\|_P^2$ being the stage cost and the terminal cost, respectively. Q, R and P are three positive definite matrices fulfilling the following conditions.

Assumption 2: Define a set $\mathbb{X}_s = \{x | \|x\|_P \leq \varepsilon\}$ and an auxiliary controller $\kappa(x)$ to fulfill the following conditions:

1) $\mathbb{X}_s \subset \mathbb{X}_f \subset \mathbb{X} \ominus \mathbb{B}(N)$, and $\kappa(x) \in \mathbb{U} \ominus \mathcal{V}$, $\forall x \in \mathbb{X}_f$;
2) $f(x) + B\kappa(x) \in \mathbb{X}_s$, $\forall x \in \mathbb{X}_f$;
3) $F(f(x) + B\kappa(x)) + l(x, \kappa(x)) - F(x) \leq 0$, $\forall x \in \mathbb{X}_f$;

Remark 3: This assumption is widely used in most MPC work with constraint tightening approach [11]. The assumption also provides a guideline to select matrices Q, R, P and design the set $\mathbb{X}_s$ and $\mathbb{X}_f$. To design the auxiliary control law $\kappa(x)$, several methods in [19], e.g., the LQR controller based on the Jacobian linearization, can be adopted.

Suppose that we have obtained the optimal control sequence $\mathbf{v}_{k_j}^*$, then the resulting optimal cost function can be denoted by $V_N^0(x_{k_j}, \mathbf{v}_{k_j}^*)$ (or $V_N^0(x_{k_j})$ for simplicity). According to (5), the actual control sequence $\mathbf{u}(k_j)$ can be generated.

Once the sampled state x_{k_j} is in $\mathbb{X}_f$, the N-step control sequence $\mathbf{v}_{k_j}$ is obtained based on $\kappa(x)$. The specific iterative processes with $i = 0, \ldots, N-1$ are given as follows,

$$\begin{aligned} v_{k_j+i|k_j} &= \kappa(\hat{x}_{k_j+i|k_j}) \\ \hat{x}_{k_j+i+1|k_j} &= f(\hat{x}_{k_j+i|k_j}) + Bv_{k_j+i|k_j}. \end{aligned} \tag{9}$$

C. Self-Triggered Mechanism

This part discusses the self-triggered mechanism. The triggering intervals are determined by the following condition:

$$m(k_j) = \begin{cases} \min\{m_f(k_j), m_s(k_j), N\} & \text{if } x(k_j) \notin \mathbb{X}_f \\ \min\{m_f(k_j), N\} & \text{if } x(k_j) \in \mathbb{X}_f \end{cases} \tag{10}$$

Algorithm 1 TDE-Based Self-Triggered MPC

Initialization: Let $j = 0$, the initial state x_{k_0}, the estimated disturbance $\hat{w}_{k_0}$, and the related parameters $N, \epsilon, \epsilon_f, P, Q, R$.

1: If $j = 0$, go to step 3. Otherwise, go to step 2;
2: At triggering instant $k = k_j$, the two states x_{k_j-1} and x_{k_j} are transmitted to obtain the estimated disturbance $\hat{w}_{k_j}$;
3: Find a $v_{k_j} \in \mathcal{V}$ to compensate $\hat{w}_{k_j}$. If $x_{k_j} \in \mathbb{X}_f$, use (9) to obtain $\mathbf{v}_{k_j}$. Otherwise, solve the OCP (8) to gain $\mathbf{v}_{k_j}$;
4: Determine the actual control sequence $\mathbf{u}_{k_j}$ and the next triggering time k_{j+1} based on (5) and (10), respectively.
5: Apply $u_{k|k_j}$ from $\mathbf{u}_{k_j}$ to system (1), and $k = k + 1$;
6: If $k = k_{j+1}$, set $j = j + 1$, and go to step 2. Otherwise, go to step 5.

where

$$\begin{aligned} m_f(k_j) = \sup\{m : L_P^{N-m+1} \sum_{s=0}^{m-1} L_P^s \min\{(m-s)\delta, \eta\} \\ \leq (\varepsilon_f - \varepsilon)/\sqrt{\bar{\lambda}(P)}\} \end{aligned} \tag{11}$$

$$\begin{aligned} m_s(k_j) = \sup\Bigg\{m :& - \sum_{k=k_j}^{k_j+m-1} \|\hat{x}_{k|k_j}\|_Q^2 + \|v_{k|k_j}^*\|_R^2 \\ &+ (\varepsilon + \varepsilon_f)\sqrt{\bar{\lambda}(P)} \sum_{s=N+1-m}^{N} L_P^s \min\{(N+1-s)\delta, \eta\} \\ &+ \sum_{i=m}^{N} \Bigg[(\sum_{s=i-m}^{i-1} \sqrt{\bar{\lambda}(Q)} L_Q^s \min\{(i-s)\delta, \eta\})^2 \\ &\quad + 2\|\hat{x}_{k_j+i|k_j}\|_Q \sqrt{\bar{\lambda}(Q)} \sum_{s=i-m}^{i-1} L_Q^s \min\{(i-s)\delta, \eta\} \Bigg] \\ &\leq -(1-\gamma) V_N^0(x_{k_j}) \Bigg\} \end{aligned} \tag{12}$$

and L_Q is given in Assumption 1 with weighted matrix Q.

Remark 4: Indeed, (11) ensures the recursive feasibility of OCP (8) and (12) is related to stability. When $x(k_j) \notin \mathbb{X}_f$, the feasibility of OCP and the decrement of optimal cost function should be guaranteed simultaneously to steer the state into $\mathbb{X}_f$. When $x(k_j) \in \mathbb{X}_f$, the feasibility condition (11) is sufficient to keep the state always in $\mathbb{X}_f$ (see Theorem 2).

Remark 5: The parameter $\gamma \in (0, 1)$ provides a trade-off between the triggering interval and the convergence speed. A larger γ usually implies a slower convergence rate but larger triggering intervals. In addition, the condition (11) is independent on the state and control sequences, leading to the periodic triggering in the terminal set $\mathbb{X}_f$.

Remark 6: Once the predictive state and control sequences have been obtained at each triggering instant, we increase m starting at 1 and check (11) and (12) to determine $m(k_j)$. Compared with (8), the computing load of such a procedure is quite low and can be neglected.

The remainder of this section concludes the self-triggered MPC scheme by Algorithm 1.

IV. Feasibility and Stability

In this section, we discuss the recursive feasibility of Algorithm 1 and the system stability.

A. Recursive Feasibility

The following lemma extended from [20, Lemma 2] is used to establish the feasibility.

Lemma 2: If $x \in \mathbb{X} \ominus \mathbb{B}(i+m)$, $y \in \mathbb{R}^n$ and $\|x-y\|_P \leq L_P^i \sum_{s=0}^{m-1} L_P^s \sqrt{\bar{\lambda}(P)} \min\{(m-s)\delta, \eta\}$, then $y \in \mathbb{X} \ominus \mathbb{B}(i)$.

Proof: Let $z = y - x + e_i$ with $e_i \in \mathbb{B}(i)$. Then $\|z\|_P \leq \|x-y\|_P + \|e_i\|_P \leq \sqrt{\bar{\lambda}(P)}(L_P^i \sum_{s=0}^{m-1} L_P^s \min\{(m-s)\delta, \eta\} + \sum_{s=0}^{i-1} L_P^s \min\{(i-s)\delta, \eta\}) \leq \sqrt{\bar{\lambda}(P)}(\sum_{s=0}^{i-1} L_P^s \min\{(m+i-s)\delta, \eta\} + \sum_{s=i}^{m+i-1} L_P^s \min\{(m+i-s)\delta, \eta\}) \leq \sum_{s=0}^{m+i-1} L_P^s \sqrt{\bar{\lambda}(P)} \min\{(m+i-s)\delta, \eta\}$, which implies that $z \in \mathbb{B}(m+i)$. By noting that $y + e_i = x + z \in \mathbb{X}$, we can conclude that $y \in \mathbb{X} \ominus \mathbb{B}(i)$. ■

Theorem 1: If Assumptions 1 and 2 hold, and OCP (8) is feasible with $x_{k_0} \in \mathbb{X}$, then the MPC with self-triggered condition (10) is recursively feasible if the disturbance satisfies

$$\min\{\delta, \eta\} \leq (\varepsilon_f - \varepsilon)/\sqrt{\bar{\lambda}(P)} L_P^N \tag{13}$$

Proof: To make the triggering condition (11) valid, i.e., the triggering interval $m(k_j), j \geq 0$ must be not less than 1, the disturbance should satisfy (13).

Suppose that OCP (8) is feasible at time k_j, then we need to prove the feasibility of the OCP (8) at k_{j+1} by constructing the following N-step control sequence

$$\bar{v}_{k|k_{j+1}} = \begin{cases} v_{k|k_j}^* & k = k_{j+1}, \ldots, k_j + N - 1 \\ \kappa(\hat{x}_{k|k_j}) & k = k_j + N \\ \kappa(\bar{x}_{k|k_{j+1}}) & k = k_j + N + 1, \ldots, k_{j+1} + N - 1 \end{cases}$$

where $\bar{x}_{k+1|k_{j+1}} = f(\bar{x}_{k|k_{j+1}}) + B\bar{v}_{k|k_{j+1}}$ with $k = k_{j+1}, \ldots, k_{j+1} + N - 1$, and $\bar{x}_{k_{j+1}|k_{j+1}} = x_{k_{j+1}}$.

In the sequel, the feasibility of the control sequence above is verified from the following four aspects:

a) $\bar{v}_{k|k_{j+1}} \in \mathbb{U} \ominus \mathcal{V}$. It directly follows from Assumption 2 and the control constraint satisfaction of $\mathbf{v}_{k_j}^*$ in OCP (8).

b) $\bar{x}_{k_{j+1}+i|k_{j+1}} \in \mathbb{X} \ominus \mathbb{B}(i)$ for all $i = 1, \ldots, N - m(k_j)$. Similar to Lemma 1, it can be easily shown that $\|\bar{x}_{k_{j+1}+i|k_{j+1}} - \hat{x}_{k_{j+1}+i|k_j}\|_P \leq \sqrt{\bar{\lambda}(P)} \sum_{s=0}^{m(k_j)-1} L_P^{i+s} \min\{(m(k_j)-s)\delta, \eta\}$. Note that $\hat{x}_{k_{j+1}+i|k_j} \in \mathbb{X} \ominus \mathbb{B}(i + m(k_j))$. Then, based on Lemma 2, we have $\bar{x}_{k_{j+1}+i|k_{j+1}} \in \mathbb{X} \ominus \mathbb{B}(i)$.

c) $\bar{x}_{k_{j+1}+i|k_{j+1}} \in \mathbb{X} \ominus \mathbb{B}(i)$ for all $i = N - m(k_j) + 1, \ldots, N-1$. We first verify that $\bar{x}_{k_j+N+1|k_{j+1}} \in \mathbb{X}_f$. In fact, since $\hat{x}_{k_j+N+1|k_j} \in \mathbb{X}_s$ and the self-triggered scheme (11), incorporating the fact that $\|\bar{x}_{k_j+N+1|k_{j+1}} - \hat{x}_{k_j+N+1|k_j}\|_P \leq L_P^{N-m(k_j)+1} \sum_{s=0}^{m(k_j)-1} L_P^s \sqrt{\bar{\lambda}(P)} \min\{(m(k_j)-s)\delta, \eta\}$, we obtain $\|\bar{x}_{k_j+N+1|k_{j+1}}\|_P \leq \|\bar{x}_{k_j+N+1|k_{j+1}} - \hat{x}_{k_j+N+1|k_j}\|_P + \|\hat{x}_{k_j+N+1|k_j}\|_P \leq \varepsilon_f - \varepsilon + \varepsilon = \varepsilon_f$. Then, based on Assumption 2, we can claim that $\bar{x}_{k_{j+1}+N-m(k_j)+1|k_{j+1}} \in \mathbb{X} \ominus \mathbb{B}(N - m(k_j))$. Furthermore, $\bar{x}_{k_{j+1}+i|k_{j+1}} \in \mathbb{X}_s \subset \mathbb{X}_f \subset \mathbb{X} \ominus \mathbb{B}(i)$ for $i = N - m(k_j) + 2, \ldots, N-1$ can also be proved.

d) $\bar{x}_{k_{j+1}+N|k_{j+1}} \in \mathbb{X}_f$. As verified above, we can directly obtain $\bar{x}_{k_{j+1}+N-1|k_{j+1}} \in \mathbb{X}_f$, it is clear that the result holds according to the second property of Assumption 2. ■

Remark 7: Compared with the most reported results, the requirement on the disturbance has been relaxed. In particular, from (13), we find that the feasibility can be guaranteed even for the large disturbance with small variation. However, large disturbance requires large feedforward compensation, which makes the control constraint in OCP (8) more rigorous.

B. Stability

In this part, we establish the stability conditions.

Theorem 2: Consider the system (1) with self-triggered scheme (10). For any state trajectories starting from $x_{k_0} \in \mathbb{X} \backslash \mathbb{X}_f$, there exists a finite time k_T (dependent on x_{k_0}) such that $x_{k_T} \in \mathbb{X}_f$ and $x_k \in \mathbb{X}_f$ for all $k \geq k_T$.

Proof: We only sketch the proof here because the details are similar to that in [11]. Note that if $V_N^0(x_{k_{j+1}}) < \gamma V_N^0(x_{k_j})$ with $0 < \gamma < 1$ holds for all $x_{k_j} \notin \mathbb{X}_f$ and $j \geq N_0$, then we can assert that the state enters $\mathbb{X}_f$ in finite time. With this fact, we can complete the proof as follows,

$$\begin{aligned} & V_N^0(x_{k_{j+1}}) - V_N^0(x_{k_j}) \\ & \leq V_N(x_{k_{j+1}}, \bar{\mathbf{v}}_{k_{j+1}}) - V_N^0(x_{k_j}) \\ & \leq -\sum_{k=k_j}^{k_j+m(k_j)-1} \|\hat{x}_{k|k_j}\|_Q^2 + \|v_{k|k_j}^*\|_R^2 \\ & \quad + \sum_{k=k_j+m(k_j)}^{k_j+N} [\|\bar{x}_{k|k_{j+1}}\|_Q^2 - \|\hat{x}_{k|k_j}\|_Q^2] \\ & \quad + l(\hat{x}_{k_j+N|k_j}, \kappa(\hat{x}_{k_j+N|k_j})) - F(\hat{x}_{k_j+N|k_j}) \\ & \quad + \sum_{k=k_j+N+1}^{k_{j+1}+N-1} l(\bar{x}_{k|k_{j+1}}, \kappa(\bar{x}_{k|k_{j+1}})) \\ & \quad + F(\bar{x}_{k_{j+1}+N|k_{j+1}}) \end{aligned} \tag{14}$$

Note that the following properties 1): $y^T\Phi y - z^T\Phi z = (y-z)^T\Phi(y+z)$, 2): $(y-z)^T\Phi(y+z) \leq (\|y\|_\Phi + \|z\|_\Phi)\|y-z\|_\Phi$, and 3): $(y-z)^T\Phi(y+z) \leq \|y-z\|_\Phi^2 + 2\|z\|_\Phi\|y-z\|_\Phi$ hold for any positive definite weighted matrix Φ. Additionally, the following two useful inequalities $F(\bar{x}_{k_{j+1}+N|k_{j+1}}) - F(\bar{x}_{k_j+N+1|k_{j+1}}) \leq -\sum_{k=k_j+N+1}^{k_{j+1}+N-1} l(\bar{x}_{k|k_{j+1}}, \kappa(\bar{x}_{k|k_{j+1}}))$ and $l(\hat{x}_{k_j+N|k_j}, \kappa(\hat{x}_{k_j+N|k_j})) \leq F(\hat{x}_{k_j+N|k_j}) - F(\hat{x}_{k_j+N+1|k_j})$ also hold because of property 3) in Assumption 2.

With the above facts and noting that $\bar{x}_{k_j+N+1|k_{j+1}} \in \mathbb{X}_f$ and $\hat{x}_{k_j+N+1|k_j} \in \mathbb{X}_s$, we obtain $V_N^0(x_{k_{j+1}}) - V_N^0(x_{k_j}) \leq \sum_{s=N+1-m(k_j)}^{N}(\varepsilon + \varepsilon_f) L_P^s \sqrt{\bar{\lambda}(P)} \min\{(N+1-s)\delta, \eta\} + \sum_{i=m(k_j)}^{N} [(\sum_{s=i-m(k_j)}^{i-1} \sqrt{\bar{\lambda}(Q)} L_Q^s \min\{(i-s)\delta, \eta\})^2 + 2\|\hat{x}_{k_j+i|k_j}\|_Q \sum_{s=i-m(k_j)}^{i-1} \sqrt{\bar{\lambda}(Q)} L_P^s \min\{(i-s)\delta, \eta\}] - \sum_{k=k_j}^{k_j+m(k_j)-1} (\|\hat{x}_{k|k_j}\|_Q^2 + \|v_{k|k_j}^*\|_R^2) \leq -(1-\gamma)V_N^0(x_{k_j})$.

It suggests that $V_N^0(x_{k_{j+1}}) < \gamma V_N^0(x_{k_j})$, which means that the system state can be steered into $\mathbb{X}_f$ in finite time.

Once the state x_{k_j} is in set $\mathbb{X}_f$, the control sequence $\mathbf{v}_{k_j}$ is obtained based on the auxiliary control law $\kappa(x)$. Then for the states during the triggering interval, we can obtain $\|x_{k_j+1} - \hat{x}_{k_j+1|k_j}\|_P \leq \cdots \leq \|x_{k_j+m(k_j)} - \hat{x}_{k_j+m(k_j)|k_j}\|_P \leq \sum_{s=0}^{m(k_j)-1} \sqrt{\bar{\lambda}(P)} L_P^s \min\{(m(k_j)-s)\delta, \eta\} \leq \varepsilon_f - \varepsilon$. Noting that $\|\hat{x}_{k_j+1|k_j}\|_P \leq \varepsilon, \ldots, \|\hat{x}_{k_j+m(k_j)|k_j}\|_P \leq \varepsilon$, then it can be verified that $\|x_{k_j+1|k_j}\|_P \leq \varepsilon_f, \ldots, \|x_{k_j+m(k_j)|k_j}\|_P \leq \varepsilon_f$, i.e., all these states will stay in $\mathbb{X}_f$. ■

V. Simulation Example

Consider the following discretized cart-and-spring system,

$$\begin{aligned} x_{1,k+1} &= x_{1,k} + T_s x_{2,k} \\ x_{2,k+1} &= (1 - T_s \tfrac{h}{M}) x_{2,k} - T_s \tfrac{k_0}{M} e^{-x_{1,k}} x_{1,k} + T_s \tfrac{u_k}{M} + w_k \end{aligned}$$

where $x_{1,k}$ and $x_{2,k}$ are the displacement and the velocity of the cart, respectively. The related parameters are provided as follows: $M = 1$ kg, $T_s = 0.2$ s, $k_0 = 0.15$ N/m, $h = 1.2$ N.s/m. The corresponding physical meanings can be found in [4]. The plant states and control inputs are restricted as $|x_1| \leq 1.8$ m,

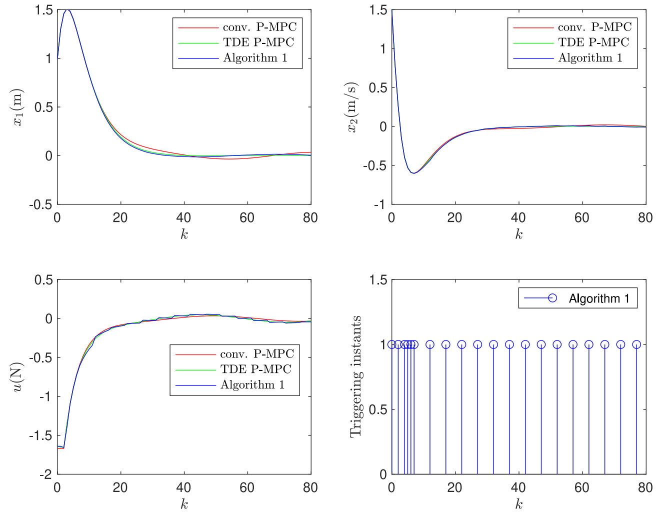

Fig. 1. Simulation results with $\eta = 0.01$ and $\delta = 0.0019$.

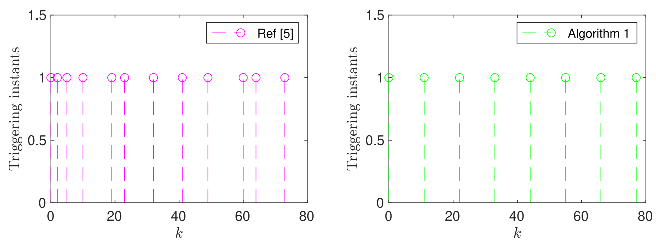

Fig. 2. Triggering instants of Algorithm 1 and the one in [5].

$|x_2| < 1.5$ m/s and $|u| < 1.7$ N. We define the quadratic stage and terminal costs, and based on the Jacobian linearization method, the associated matrices, auxiliary control law and sets are selected as $Q = [0.1, 0; 0, 0.1]$, $R = 0.1$, $P = [2.8111, 1.0972; 1.0972, 1.1884]$, $\kappa(x) = [-0.7489 - 0.9535]x$, $\mathbb{X}_f = \{x : x^T Px \leq 2.9\}$, and $\mathbb{X}_s = \{x : x^T Px \leq 2.5804\}$, respectively. The prediction horizon N is set as 12. The initial state is $x_0 = [1.0, 1.5]^T$. The OCP in (8) can be solved by using the MATLAB function `fmincon`.

When the disturbance satisfies $\eta = 0.01$ and $\delta = 0.0019$, the step length N_c is 5. Simulation results, including the state responses, the control inputs and the triggering times, are all illustrated in Fig. 1. To exhibit the superiority of Algorithm 1, we conduct a comparison of conventional periodic MPC (conv. P-MPC) and the TDE-based periodic MPC (TDE P-MPC, i.e., the conv. P-MPC with feedforward compensation (5)). Note that all these MPC schemes achieve the constraints satisfaction and stability. We also observe that the disturbance can be effectively compensated by the TDE-based MPC. Moreover, compared with the TDE P-MPC, our algorithm reduces the transmissions as well as the frequency of solving OCP significantly, and has a comparable control performance.

Notice that when $\eta = 0.01$, the event-/self-triggered MPC algorithms in [5], [11] fails to work since the recursive feasibility cannot be guaranteed. Therefore, we consider the case when $\eta = 0.0042$ and $\delta = 0.001$ (i.e., $N_c = 4$), and compare our algorithm with the one in [5]. The results are shown in Fig. 2, validating the superior of our algorithm. The triggering caused by Algorithm 1 is periodic because δ is so small that the condition in (12) always holds.

VI. Conclusion

This brief has proposed a TDE-based self-triggered MPC scheme by considering more information on the disturbance, with discussions on the feasibility and stability. The unnecessary transmissions and the frequency of solving the MPC have been reduced. Extending the proposed scheme to handle unmatched disturbances will be one of our future endeavors.

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