Constrained Common Invariant Subspace and Its Application

Dongdong Zhao, *Member, IEEE*, Yu Kang, *Senior Member, IEEE*, Yun-Bo Zhao, *Senior Member, IEEE*, Li Xu, *Senior Member, IEEE*, and Shi Yan, *Member, IEEE*

***Abstract*—The notion of constrained common invariant subspaces (CCISs) is proposed in this article as a generalization of the well-known invariant subspace to study the structural properties of multiple matrices. Specifically, some necessary and sufficient conditions for the existence of a CCIS are established to provide a methodology to compute such a CCIS. Then, the properties of CCISs and their relation to common eigenvectors are revealed. The existence of common eigenvectors leads to the existence of CCIS, but not vice versa, so the established CCIS can reveal the structural properties of multiple matrices better than common eigenvectors can. The established CCIS is applied to the reducibility of Fornasini–Marchesini (F-M) state-space models, i.e., the necessary and sufficient conditions and the related algorithm for reducibility of F-M models are developed. Finally, a gain-scheduled state-feedback control is proposed for a rational parameter system to further demonstrate the superiority of the established CCIS.**

***Index Terms*—Invariant subspace (IS), order reduction, rational parameter system, state-feedback control.**

I. Introduction

THE invariant subspace (IS) of a single square matrix has been extensively studied and applied as the so-called geometric control theory for conventional linear time-invariant systems [1], [2], [3]. The IS has found application in system identification, analysis, and design, e.g., identification [3], [4], reducibility (or minimality) [5], [6], controllability analysis [7], observability analysis [8], linear-quadratic mean field control [9], and state feedback and observer design [10]. An essential application of the IS is to clarify the internal relationship between reducibility (or minimality) and controllability/observability in the framework of the geometric approach [6]. Consider a linear system represented by

$$x(i+1) = Ax(i) + Bu(i),\ x \in \mathbb{R}^r,\ u \in \mathbb{R}^q \qquad (1a)$$

$$y(i) = Cx(i) + Du(i),\ y \in \mathbb{R}^p. \qquad (1b)$$

If there exists an IS $\mathcal{W}$ under the state matrix A satisfying $C\boldsymbol{w} = 0$ for every $\boldsymbol{w} \in \mathcal{W}$, then the linear system of (1) is not observable and can be exactly reduced [11]. Other concepts and their applications related to the IS, such as controlled IS and conditioned IS, have been discussed in several books (e.g., [6], [12]).

However, in many practical complex systems, the dynamic state vector of a system is related to multiple matrices. For example, the Fornasini–Marchesini (F-M) (state-space) models, in contrast to traditional linear state-space models, propagate information in n different ($n \geq 2$) independent directions and correspondingly have n state-space matrices $A_1, \ldots, A_n \in \mathbb{R}^{r\times r}$ [13], [14], [15]. The F-M has received considerable attention because it can model many practical systems, e.g., river pollution modeling [16] and distributed grid sensor networks [17]. Linear parameter-varying (LPV) systems are another important system class [18], [19], where the dynamic state vector is related to multiple matrices A_i. Specifically, the parameter-varying state transition can be expressed as $\mathcal{A}(\theta) = \sum_{i=1}^{n} A_i\theta_i$ where θ_i denotes the time-varying parameters.

Therefore, to study the structural properties of multiple square matrices, the generalized common IS (CIS) has recently attracted interest [20], [21], [22], [23]. Pastuszak revealed in [22] that CIS plays an important role in quantifier elimination theory. Arroyo et al. showed in [20] that CIS can overcome the challenges inherent in appropriately modeling graph differences while retaining sufficient model simplicity to render estimation feasible. However, the existence conditions and computation of CIS remain a difficult problem [22], [23] because even generalizing the results of IS to CIS is a long-standing goal and challenge for researchers. Considerable effort has been made to address this difficult problem. In [24], Shemesh presented a computable condition for the existence of a common eigenvector (CE), i.e., a special case of CIS with dimension one [25], of two matrices, which is generalized to a finite number of matrices in [26]. In [27], although the case of two matrices is considered, Al'pin and Ikramov studied the CIS of dimensions greater than one;

Manuscript received 6 December 2023; revised 1 March 2024; accepted 14 March 2024. Date of publication 20 March 2024; date of current version 27 September 2024. This work was supported in part by the National Key Research and Development Program of China under Grant 2018AAA0100801, in part by the National Natural Science Foundation of China under Grant U22B2040, Grant 62233003, Grant 62033012, and Grant 62173317, and in part by the Natural Science Foundation of Gansu Province of China under Grant 20JR10RA638. Recommended by Associate Editor Z. Shu. *(Corresponding author: Shi Yan.)*

Dongdong Zhao and Shi Yan are with the School of Information Science and Engineering, Lanzhou University, Lanzhou 73000, China (e-mail: ddzhao@lzu.edu.cn; yanshi@lzu.edu.cn).

Yu Kang and Yun-Bo Zhao are with the Department of Automation, University of Science and Technology of China, Hefei 230027, China (e-mail: kangduyu@ustc.edu.cn; ybzhao@ustc.edu.cn).

Li Xu is with the Department of Electronics and Information Systems, Akita Prefectural University, Akita 015-0055, Japan (e-mail: xuli@akita-pu.ac.jp).

Digital Object Identifier 10.1109/TAC.2024.3378769

however, they assumed that given matrices have pairwise different eigenvalues, as noted in [25]. Arapura and Peterson in [23] provided a general existence condition of CIS for an arbitrary finite number of matrices by converting the computation of CIS to find the projective variety of a series of symbolic equations by applying the Gröbner basis technique, which requires knowledge of the dimensionality of CIS. Moreover, the Gröbner basis technique is useful for solving polynomial equations [28], but it is time-consuming and difficult to obtain solutions for a large number of variables [29]. Consequently, applying the Gröbner basis method to large-scale matrices is a challenging problem.

One approach to partially avoid this difficulty is to study the CIS with certain appropriate constraints. To this end, we recently established a notion of constrained CE (CCE) in [30], i.e., the 1-D CIS with a basis vector $\boldsymbol{w}$ satisfying $C\boldsymbol{w} = \mathbf{0}$. A necessary and sufficient condition for the existence of a CCE was also given in [30]. However, the existence conditions and computation of the more general CIS with constraints and dimensions greater than one remain unsolved.

Given the above background, this article aims to establish a constrained CIS approach such that the structural properties of all the multiple square matrices can be explored. Specifically, a new notion called constrained CIS (CCIS) is presented, and the necessary and sufficient conditions for the existence of a CCIS are established. Then, the properties of CCIS and its relation to CE are revealed. The existence of a CE leads to the existence of a CCIS, but not vice versa, so the established CCIS can reveal the structural properties of multiple matrices better than the CE can. Based on the advantages of CCIS to reveal the structural characteristics of multiple matrices, it has been applied to F-M models, where the sufficient and necessary conditions and the related algorithms are developed for the reducibility of the F-M models. Moreover, gain-scheduled state-feedback control is proposed for a rational parameter system, the numerical complexity of which can be greatly reduced by the established CCIS.

The rest of the article is organized as follows. In Section II, the notion of CCIS is given; then, the necessary and sufficient conditions for the existence of CCIS are established. Section III reveals some properties of CCIS. Section IV applies the established CCIS to F-M models and presents sufficient and necessary reducibility conditions for the F-M models. Section V proposes an H_∞ gain-scheduled state-feedback controller synthesis for rational parameter systems. Finally, Section VI concludes this article.

The following notations will be adopted in this article. The r-dimensional real column and row vector spaces are denoted by $\mathbb{R}^r$ and $\mathbb{R}^{1\times r}$, respectively. Let $\mathbb{R}(\theta_k)$ be the field of rational functions in variables $\theta_{k1}, \ldots, \theta_{kn}$ over $\mathbb{R}$, $\mathbb{R}^{m\times l}$ be the class of $m \times l$ matrices with entries in $\mathbb{R}$, and $\mathbb{R}^{m\times l}(\theta_k)$ be the class of $m \times l$ matrices with entries in $\mathbb{R}(\theta_k)$. We denote the positive integers by $\mathbb{Z}_+$. For a number $n \in \mathbb{Z}_+$, let $\mathcal{N}$ represent the set $\{1, \ldots, n\}$. Denote by $\dim(\mathcal{W})$ the dimensionality of the space $\mathcal{W}$. A zero matrix with proper size and a matrix of specified size $p \times q$ are correspondingly denoted by the symbols $\mathbf{0}$ and $\mathbf{0}_{p\times q}$. For a matrix $\mathcal{M} \in \mathbb{R}^{p\times q}$, let $\mathrm{span}_{\mathrm{row}}(\mathcal{M})$ denote the space spanned by the row vectors of $\mathcal{M}$; the space spanned by the vectors $\boldsymbol{v}_1, \ldots, \boldsymbol{v}_r$ is denoted by $\mathrm{span}(\boldsymbol{v}_1, \ldots, \boldsymbol{v}_r)$; the kernel of $\mathcal{M}$ is $\ker(\mathcal{M}) = \{\boldsymbol{\eta} \in \mathbb{R}^q : \mathcal{M}\boldsymbol{\eta} = \mathbf{0}\}$. $A^\top$ stands for the transpose of A. $\{A\}^s$ represents the sum of A and its transpose, i.e., $\{A\}^s := A + A^\top$.

II. Constrained Common IS

In this section, the notion of CCIS will be generalized from the notions of CIS [22] and CCE [30]. Then, the conditions for the existence of CCIS, which provide a technique computing a CCIS and play an essential role in studying the structural properties of multiple square matrices, will be developed.

Let $A_1, \ldots, A_n \in \mathbb{R}^{r\times r}$, $B \in \mathbb{R}^{r\times q}$, $C \in \mathbb{R}^{p\times r}$, and $\mathcal{A} = \{A_1, \ldots, A_n\}$. We present the following notions.

Definition 1: A subspace $\mathcal{W} \subseteq \mathbb{R}^r$ is called a right CIS of $A_1, \ldots, A_n$ if

$$A_1\boldsymbol{w} \in \mathcal{W} \tag{2a}$$

$$\vdots \tag{2b}$$

$$A_n\boldsymbol{w} \in \mathcal{W} \tag{2c}$$

for all $\boldsymbol{w} \in \mathcal{W}$ [22]. Dually, $\mathcal{W} \subseteq \mathbb{R}^r$ is called a left CIS of $A_1, \ldots, A_n$ if

$$\boldsymbol{w}^\top A_1 \in \mathcal{W} \tag{3a}$$

$$\vdots \tag{3b}$$

$$\boldsymbol{w}^\top A_n \in \mathcal{W} \tag{3c}$$

for all $\boldsymbol{w} \in \mathcal{W}$.

Definition 2: A right CIS $\mathcal{W}$ of $A_1, \ldots, A_n$ is said to be constrained by C if $C\boldsymbol{w} = \mathbf{0}$ for all $\boldsymbol{w} \in \mathcal{W}$. Dually, a left CIS $\mathcal{W}$ of $A_1, \ldots, A_n$ is said to be constrained by B if $\boldsymbol{w}^\top B = \mathbf{0}$ for all $\boldsymbol{w} \in \mathcal{W}$. If it is not necessary to indicate the associated matrices explicitly, such a subspace will be referred to as a right/left constrained CIS or CCIS. A right/left CCIS $\mathcal{W}$ is called trivial if $\dim(\mathcal{W}) = 0$; otherwise, it is called nontrivial.

Definition 3: The matrix functions $\mathcal{L}_k(\mathcal{A}, C)$ and $\mathcal{M}_k(\mathcal{A}, C)$ are recursively defined by

$$\mathcal{L}_k(\mathcal{A}, C) := \begin{bmatrix} \mathcal{M}_{k-1}(\mathcal{A}, C)A_1 \\ \vdots \\ \mathcal{M}_{k-1}(\mathcal{A}, C)A_n \end{bmatrix} \in \mathbb{R}^{pn^{k-1}\times r} \tag{4a}$$

$$\mathcal{M}_k(\mathcal{A}, C) := \begin{bmatrix} \mathcal{L}_1(\mathcal{A}, C) \\ \vdots \\ \mathcal{L}_k(\mathcal{A}, C) \end{bmatrix} \in \mathbb{R}^{\frac{p\left(1-n^k\right)}{1-n}\times r} \tag{4b}$$

with $\mathcal{M}_1(\mathcal{A}, C) = \mathcal{L}_1(\mathcal{A}, C) = C$ and $k = 2, \ldots, \infty$.

The existence conditions of a right CCIS can be formulated to find a kernel of an infinite matrix as follows.

Theorem 1: There exists a nontrivial right CIS $\mathcal{W}$ of $A_1, \ldots, A_n \in \mathbb{R}^{r\times r}$ constrained by $C \in \mathbb{R}^{p\times r}$ if and only if $\mathcal{M}_\infty(\mathcal{A}, C)$ is rank deficient. Moreover

$$\mathcal{W} = \ker\left(\mathcal{M}_\infty(\mathcal{A}, C)\right) \tag{5}$$

with

$$\mathcal{M}_\infty(\mathcal{A}, C) = \lim_{k\to\infty} \mathcal{M}_k(\mathcal{A}, C).$$

Proof: We first prove the necessity and then sufficiency. Assume that $\mathcal{W}$ is a nontrivial right CCIS. Thus, dim $(\mathcal{W}) \geq 1$, and for any $\boldsymbol{w} \in \mathcal{W}$, we have

$$A_i\boldsymbol{w} \in \mathcal{W} \tag{6a}$$

$$C\boldsymbol{w} = 0. \tag{6b}$$

According to (6) and Definition 3, we derive

$$\mathcal{M}_\infty(\mathcal{A}, C)\boldsymbol{w} = 0 \tag{7}$$

which gives the rank deficiency of $\mathcal{M}_\infty(\mathcal{A}, C)$.

To show sufficiency, assume that $\mathcal{M}_\infty(\mathcal{A}, C)$ is rank deficient. Let

$$\mathcal{W} := \ker\left(\mathcal{M}_\infty(\mathcal{A}, C)\right) \neq \emptyset. \tag{8}$$

Then, for any $\boldsymbol{w} \in \mathcal{W}$, we deduce

$$\mathcal{M}_\infty(\mathcal{A}, C)\boldsymbol{w} = \mathbf{0}. \tag{9}$$

By the definition of $\mathcal{M}_k(\mathcal{A}, C)$ and $\mathcal{L}_k(\mathcal{A}, C)$ in (4), we derive that

$$\mathcal{M}_k(\mathcal{A}, C) = \begin{bmatrix} \mathcal{M}_{k-1}(\mathcal{A}, C) \\ \mathcal{M}_{k-1}(\mathcal{A}, C)A_1 \\ \vdots \\ \mathcal{M}_{k-1}(\mathcal{A}, C)A_n \end{bmatrix}, \ k = 2, \ldots, \infty. \tag{10}$$

$\mathcal{M}_k(\mathcal{A}, C)\boldsymbol{w} = \mathbf{0}$ and (10) imply that

$$\mathcal{M}_{k-1}(\mathcal{A}, C)A_i\boldsymbol{w} = \mathbf{0} \text{ for all } i = 1, \ldots, n.$$

Then, we derive from (9) that

$$\mathcal{M}_k(\mathcal{A}, C)A_i\boldsymbol{w} = \mathbf{0} \text{ for all } k = 1, \ldots, \infty, i = 1, \ldots, n \tag{11}$$

which indicates $A_i\boldsymbol{w} \in \mathcal{W}$. In view of $\mathcal{M}_\infty(\mathcal{A}, C)$ containing $\mathcal{M}_1(\mathcal{A}, C) = C$, we obtain

$$C\boldsymbol{w} = \mathcal{M}_1(\mathcal{A}, C)\boldsymbol{w} = \mathbf{0}. \tag{12}$$

Therefore, by (12) and Definition 2, we conclude that $\mathcal{W}$ is a nontrivial right CCIS. ■

Lemma 1: If $\mathrm{span}_{\mathrm{row}}(\mathcal{M}_k(\mathcal{A}, C) = \mathrm{span}_{\mathrm{row}}(\mathcal{M}_{k+1}(\mathcal{A}, C))$, then $\mathrm{span}_{\mathrm{row}}(\mathcal{M}_k(\mathcal{A}, C)) = \mathrm{span}_{\mathrm{row}}(\mathcal{M}_{k+2}(\mathcal{A}, C))$.

Proof: Suppose that

$$\mathrm{span}_{\mathrm{row}}\left(\mathcal{M}_{k+1}(\mathcal{A}, C)\right) = \mathrm{span}_{\mathrm{row}}\left(\mathcal{M}_k(\mathcal{A}, C)\right). \tag{13}$$

By Definition 3, we derive that

$$\begin{aligned} &\mathrm{span}_{\mathrm{row}}\left(\mathcal{M}_{k+1}(\mathcal{A}, C)\right) \\ &\quad = \mathrm{span}_{\mathrm{row}}\left(\mathcal{M}_k(\mathcal{A}, C)\right) + \mathrm{span}_{\mathrm{row}}\left(\mathcal{L}_{k+1}(\mathcal{A}, C)\right) \\ &\quad = \mathrm{span}_{\mathrm{row}}\left(\mathcal{M}_k(\mathcal{A}, C)\right) + \mathrm{span}_{\mathrm{row}}\left(\mathcal{M}_k(\mathcal{A}, C)A_1\right) \\ &\qquad + \cdots + \mathrm{span}_{\mathrm{row}}\left(\mathcal{M}_k(\mathcal{A}, C)A_n\right). \end{aligned} \tag{14}$$

It follows from (13) and (14) that:

$$\mathrm{span}_{\mathrm{row}}\left(\mathcal{M}_k(A, C)A_i\right) \subseteq \mathrm{span}_{\mathrm{row}}\left(\mathcal{M}_k(\mathcal{A}, C)\right) \tag{15}$$

which indicates that

$$\boldsymbol{w}^{\mathrm{T}}A_i \in \mathrm{span}_{\mathrm{row}}\left(\mathcal{M}_k(\mathcal{A}, C)\right), \ i = 1, \ldots, n \tag{16}$$

for all $\boldsymbol{w} \in \mathrm{span}_{\mathrm{row}}(\mathcal{M}_k(\mathcal{A}, C)) = \mathrm{span}_{\mathrm{row}}(\mathcal{M}_{k+1}(\mathcal{A}, C))$. Then

$$\begin{aligned} &\mathrm{span}_{\mathrm{row}}\left(\mathcal{L}_{k+2}(\mathcal{A}, C)\right) \\ &\quad = \mathrm{span}_{\mathrm{row}}\left(\begin{bmatrix} \mathcal{M}_{k+1}(\mathcal{A}, C)A_1 \\ \vdots \\ \mathcal{M}_{k+1}(\mathcal{A}, C)A_n \end{bmatrix}\right) \\ &\quad \subseteq \mathrm{span}_{\mathrm{row}}\left(\mathcal{M}_{k+1}(\mathcal{A}, C)\right). \end{aligned} \tag{17}$$

According to Definition 3, we have

$$\begin{aligned} &\mathrm{span}_{\mathrm{row}}\left(\mathcal{M}_{k+2}(\mathcal{A}, C)\right) \\ &\quad = \mathrm{span}_{\mathrm{row}}\left(\mathcal{M}_{k+1}(\mathcal{A}, C)\right) + \mathrm{span}_{\mathrm{row}}\left(\mathcal{L}_{k+2}(\mathcal{A}, C)\right). \end{aligned} \tag{18}$$

Combining (18) with (17) gives

$$\mathrm{span}_{\mathrm{row}}\left(\mathcal{M}_{k+2}(\mathcal{A}, C)\right) = \mathrm{span}_{\mathrm{row}}\left(\mathcal{M}_{k+1}(\mathcal{A}, C)\right). \tag{19}$$

We conclude from (13) and (19) that $\mathrm{span}_{\mathrm{row}}(\mathcal{M}_{k+2}(\mathcal{A}, C)) = \mathrm{span}_{\mathrm{row}}(\mathcal{M}_k(\mathcal{A}, C))$. ■

Lemma 2: $\mathrm{span}_{\mathrm{row}}(\mathcal{M}_r(\mathcal{A}, C)) = \mathrm{span}_{\mathrm{row}}(\mathcal{M}_\infty(\mathcal{A}, C))$.

Proof: It is noted that if $C = \mathbf{0}$, we can easily from Definition 3 conclude that

$$\begin{aligned} &\mathrm{span}_{\mathrm{row}}\left(\mathcal{M}_1(\mathcal{A}, C)\right) = \mathrm{span}_{\mathrm{row}}\left(\mathcal{M}_2(\mathcal{A}, C)\right) = \cdots \\ &\quad = \mathrm{span}_{\mathrm{row}}\left(\mathcal{M}_r(\mathcal{A}, C)\right) = \cdots \\ &\quad = \mathrm{span}_{\mathrm{row}}\left(\mathcal{M}_\infty(\mathcal{A}, C)\right) = \{\mathbf{0}\}. \end{aligned} \tag{20}$$

Thus, we only need to show the case of $C \neq \mathbf{0}$.

By Definition 3, we have

$$\begin{aligned} &\mathrm{span}_{\mathrm{row}}\left(\mathcal{M}_k(\mathcal{A}, C)\right) \\ &\quad = \mathrm{span}_{\mathrm{row}}\left(\mathcal{M}_{k-1}(\mathcal{A}, C)\right) + \mathrm{span}_{\mathrm{row}}\left(\mathcal{L}_k(\mathcal{A}, C)\right) \end{aligned} \tag{21}$$

which indicates

$$\mathrm{span}_{\mathrm{row}}\left(\mathcal{M}_{k-1}(\mathcal{A}, C)\right) \subseteq \mathrm{span}_{\mathrm{row}}\left(\mathcal{M}_k(\mathcal{A}, C)\right) \tag{22}$$

for $k = 1, \ldots, \infty$. Thus

$$\begin{aligned} &\mathrm{span}_{\mathrm{row}}\left(\mathcal{M}_1(\mathcal{A}, C)\right) \subseteq \mathrm{span}_{\mathrm{row}}\left(\mathcal{M}_2(\mathcal{A}, C)\right) \\ &\quad \subseteq \cdots \subseteq \mathrm{span}_{\mathrm{row}}\left(\mathcal{M}_\infty(\mathcal{A}, C)\right). \end{aligned} \tag{23}$$

From (23) and Lemma 1, we deduce that

$$\begin{aligned} &\mathrm{span}_{\mathrm{row}}\left(\mathcal{M}_1(\mathcal{A}, C)\right) \subset \mathrm{span}_{\mathrm{row}}\left(\mathcal{M}_2(\mathcal{A}, C)\right) \subset \cdots \\ &\quad \subset \mathrm{span}_{\mathrm{row}}\left(\mathcal{M}_l(\mathcal{A}, C)\right) = \mathrm{span}_{\mathrm{row}}\left(\mathcal{M}_{l+1}(\mathcal{A}, C)\right) = \cdots \\ &\quad = \mathrm{span}_{\mathrm{row}}\left(\mathcal{M}_\infty(\mathcal{A}, C)\right) \end{aligned} \tag{24}$$

for some $l \in \{1, \ldots, \infty\}$, which gives

$$\begin{aligned} &\mathrm{rank}\left(\mathcal{M}_1(\mathcal{A}, C)\right) < \mathrm{rank}\left(\mathcal{M}_2(\mathcal{A}, C)\right) < \cdots \\ &\quad < \mathrm{rank}\left(\mathcal{M}_l(\mathcal{A}, C)\right) = \mathrm{rank}\left(\mathcal{M}_{l+1}(\mathcal{A}, C)\right) = \cdots \\ &\quad = \mathrm{rank}\left(\mathcal{M}_\infty(\mathcal{A}, C)\right). \end{aligned} \tag{25}$$

Since $\mathcal{M}_1(\mathcal{A}, C) = C \neq \mathbf{0}$, we have

$$\mathrm{rank}\left(\mathcal{M}_1(\mathcal{A}, C)\right) \geq 1. \tag{26}$$

Algorithm 1: Computation of a Right CCIS.

Input: $A_1, \ldots, A_n \in \mathbb{R}^{r\times r}$, $C \in \mathbb{R}^{p\times r}$.
Output: $\mathcal{W}$.

1 $\mathcal{M}_1(\mathcal{A}, C) \leftarrow C$;
2 **for** $k = 2, \ldots, r$ **do**
3 Compute $\mathcal{L}_k(\mathcal{A}, C)$ as in (4a);
4 Compute $\mathcal{M}_k(\mathcal{A}, C)$ using (4b);
5 $\mathcal{W} \leftarrow \ker\left(\mathcal{M}_r(\mathcal{A}, C)\right)$.

From (25) and (26), it follows that

$$\begin{aligned}\operatorname{rank}\left(\mathcal{M}_2(\mathcal{A}, C)\right) &\geq 2\\ \operatorname{rank}\left(\mathcal{M}_3(\mathcal{A}, C)\right) &\geq 3\\ &\vdots\\ \operatorname{rank}\left(\mathcal{M}_l(\mathcal{A}, C)\right) &\geq l. \end{aligned} \tag{27}$$

On the other hand, it can be seen from Definition 3 that the matrix $\mathcal{M}_l(\mathcal{A}, C)$ has r columns, i.e.,

$$\operatorname{rank}\left(\mathcal{M}_l(\mathcal{A}, C)\right) \leq r \tag{28}$$

According to (27) and (28), we have

$$l \leq \operatorname{rank}\left(\mathcal{M}_l(\mathcal{A}, C)\right) \leq r \tag{29}$$

which indicates

$$l \leq r. \tag{30}$$

Therefore, it can be obtained from (24) and (30) that

$$\begin{aligned}\operatorname{span}_{\text{row}}\left(\mathcal{M}_r(\mathcal{A}, C)\right) &= \operatorname{span}_{\text{row}}\left(\mathcal{M}_{r+1}(\mathcal{A}, C)\right) = \cdots\\ &= \operatorname{span}_{\text{row}}\left(\mathcal{M}_{\infty}(\mathcal{A}, C)\right).\end{aligned}$$

■

The existence condition of a right CCIS given by Theorem 1 depends on an infinite matrix and can be converted equivalently into the following one, which is described by a finite-dimensional matrix.

Theorem 2: There exists a nontrivial right CCIS $\mathcal{W}$ of $A_1, \ldots, A_n \in \mathbb{R}^{r\times r}$ constrained by $C \in \mathbb{R}^{p\times r}$ if and only if $\mathcal{M}_r(\mathcal{A}, C)$ is rank deficient. Moreover

$$\mathcal{W} = \ker\left(\mathcal{M}_r(\mathcal{A}, C)\right). \tag{31}$$

Proof: It directly follows from the results of Lemma 2 and Theorem 1. ■

Theorems 1 and 2 represent an algorithm to derive a right CIS of $A_1, \ldots, A_n \in \mathbb{R}^{r\times r}$ constrained by $C \in \mathbb{R}^{p\times r}$, described by the pseudocode in Algorithm II.

Remark 1: Note that Algorithm II starts by initializing matrix $\mathcal{M}_1(\mathcal{A}, C)$ as C. Then, the algorithm enters the for-loop on line 2. Within this loop, from line 3 to line 4, the matrices $\mathcal{L}_k(\mathcal{A}, C)$ and $\mathcal{M}_k(\mathcal{A}, C)$ are recursively obtained, $k = 2, \ldots, r$. Finally, $\mathcal{W}$ is computed as the kernel space of $\mathcal{M}_r(\mathcal{A}, C)$. Theorem 2 shows that the kernel space of $\mathcal{M}_r(\mathcal{A}, C)$ is a right CCIS of $A_1, \ldots, A_n$ constrained by C. Thus, the obtained $\mathcal{W}$ is the desired right CCIS.

Example 1: To demonstrate the specifics and effectiveness of Algorithm II, consider the following matrices:

$$A_1 = \begin{bmatrix} 0 & 0 & 1 & 1\\ 2 & 0 & 1 & -1\\ 0 & 1 & 0 & -1\\ -1 & 2 & 3 & 2\end{bmatrix}, \; A_2 = \begin{bmatrix} 2 & 1 & 0 & 0\\ -2 & 2 & 1 & 1\\ 1 & -3 & -2 & 0\\ 1 & 2 & 1 & 1\end{bmatrix}$$
$$C = \begin{bmatrix} 1 & 1 & 1 & -1\end{bmatrix}. \tag{32}$$

By means of Algorithm II, we obtain

$$\begin{aligned}&\mathcal{M}_4(\mathcal{A}, C)\\ &= \begin{bmatrix}\mathcal{L}_1^{\top}(\mathcal{A}, C) & \mathcal{L}_2^{\top}(\mathcal{A}, C) & \mathcal{L}_3^{\top}(\mathcal{A}, C) & \mathcal{L}_4^{\top}(\mathcal{A}, C)\end{bmatrix}^{\top}\end{aligned} \tag{33}$$

with

$$\mathcal{L}_1(\mathcal{A}, C) = C = \begin{bmatrix} 1 & 1 & 1 & -1\end{bmatrix}$$

$$\mathcal{L}_2(\mathcal{A}, C) = \begin{bmatrix} CA_1\\ CA_2\end{bmatrix} = \begin{bmatrix} 3 & -1 & -1 & -3\\ 0 & -2 & -2 & 0\end{bmatrix}$$

$$\mathcal{L}_3(\mathcal{A}, C) = \begin{bmatrix} CA_1A_1\\ CA_2A_1\\ CA_1A_2\\ CA_2A_2\end{bmatrix} = \begin{bmatrix} 1 & -7 & -7 & -1\\ -4 & -2 & -2 & 4\\ 4 & -2 & -2 & -4\\ 2 & 2 & 2 & -2\end{bmatrix}$$

$$\mathcal{L}_4(\mathcal{A}, C) = \begin{bmatrix} CA_1A_1A_1\\ CA_2A_1A_1\\ CA_1A_2A_1\\ CA_2A_2A_1\\ CA_1A_1A_2\\ CA_2A_1A_2\\ CA_1A_2A_2\\ CA_2A_2A_2\end{bmatrix} = \begin{bmatrix} -13 & -9 & -9 & 13\\ -8 & 6 & 6 & 8\\ 0 & -10 & -10 & 0\\ 6 & -2 & -2 & -6\\ 8 & 6 & 6 & -8\\ -2 & 6 & 6 & 2\\ 6 & -2 & -2 & -6\\ 0 & -4 & -4 & 0\end{bmatrix}.$$

The kernel of $\mathcal{M}_4(\mathcal{A}, C)$ is

$$\begin{aligned}\mathcal{W} &= \ker\left(\mathcal{M}_4(\mathcal{A}, C)\right) = \operatorname{span}\{\boldsymbol{v}_1, \boldsymbol{v}_2\}\\ &= \operatorname{span}\left\{\begin{bmatrix} 0\\ -1\\ 1\\ 0\end{bmatrix}, \begin{bmatrix} 1\\ 0\\ 0\\ 1\end{bmatrix}\right\}\end{aligned} \tag{34}$$

which is a right CIS of A_1 and A_2 constrained by C.

Let $\mathcal{A}^{\top} := \{A_1^{\top}, \ldots, A_n^{\top}\}$. By the duality of left and right CCISs, the condition for the existence of the left CCIS can be given as follows.

Theorem 3: There exists a nontrivial left CIS $\mathcal{W}$ of $A_1, \ldots, A_n \in \mathbb{R}^{r\times r}$ constrained by $B \in \mathbb{R}^{r\times q}$ if and only if $\mathcal{M}_r(\mathcal{A}^{\top}, B^{\top})$ is rank deficient. Moreover

$$\mathcal{W} = \ker\left(\mathcal{M}_r(\mathcal{A}^{\top}, B^{\top})\right). \tag{35}$$

Since Theorem 3 can be proved similarly to Theorem 2, the details are omitted here for the sake of brevity.

III. Properties of CCIS

In the previous section, the existence conditions of the right and left CCISs were given based on the ranks of the related matrices. In this section, the existence of a right and left CCIS will be incorporated into a single theorem. Then, the relationship between CCIS and CE will be revealed.

A. Existence Conditions of CCISs

Definition 4: For a given $n \in \mathbb{Z}_+$, $\mathcal{F}_\mathcal{N}$ denotes the set consisting of sequences of elements in $\mathcal{N} := \{1, \dots, n\}$. The elements of $\mathcal{F}_\mathcal{N}$ are also called strings or words [31]. Each $v \in \mathcal{F}_\mathcal{N}$ is in the form of

$$v = \alpha_1 \alpha_2 \cdots \alpha_l$$

for some $\alpha_k \in \mathcal{N}$, $k = 1, \dots, l$, where α_k stands for the kth letter of v and $l = |v|$ is the length of v. Let ϵ denote an empty word, and $|\epsilon| = 0$. We denote by $\mathcal{F}_\mathcal{N}^+$ the set of nonempty words.

Definition 5: For a finite collection of matrix $A_i \in \mathbb{R}^{n\times n}$, $i \in \mathcal{N}$ and an element $v = \alpha_1\alpha_2\cdots\alpha_l \in \mathcal{F}_\mathcal{N}$, $A^{(v)}$ is defined by

$$A^{(v)} = A_{\alpha_l} A_{\alpha_l - 1} \cdots A_{\alpha_1} \in \mathbb{R}^{n\times n} \tag{36}$$

where $A^{(v)} = I$ if $v = \epsilon$, i.e., $A_\epsilon = I$.

Definition 6 (Lexicographic Ordering): The lexicographic ordering $<$ on $\mathcal{F}_\mathcal{N}$ is defined as follows [28]. For any $v_1, v_2 \in \mathcal{F}_\mathcal{N}$ with $v_1 = \alpha_1\,\alpha_2\,\cdots\,\alpha_{l_1}$ and $v_2 = \beta_1\,\beta_2\,\cdots\,\beta_{l_2}$, $v_1 < v_2$ if either $|v_1| < |v_2|$, i.e., $l_1 < l_2$ or $0 < |v_1| = |v_2|$, $v_1 \neq v_2$ and for some $k \in \{1, \dots, |v_1|\}$, $\alpha_k < \beta_k$ with the usual ordering of integers and $\alpha_i = \beta_i$ for $i = 1, \dots, k-1$. With lexicographic ordering, the elements of

$$\mathcal{F}_\mathcal{N} = \{v_1,\ v_2, \dots\} \tag{37}$$

have the relationship of $\epsilon = v_1 < v_2 < \cdots$. Notably, the lexicographic ordering is a complete ordering on the set $\mathcal{F}_\mathcal{N}$ [32]. In other words, $v_1 < v_2$ implies $v_3v_1v_4 < v_3v_2v_4$ for all $v_1, v_2, v_3, v_4 \in \mathcal{F}_\mathcal{N}\backslash\{\epsilon\}$, where $\mathcal{F}_\mathcal{N}\backslash\{\epsilon\} := \{v : v \in \mathcal{F}_\mathcal{N}$ and $v \notin \{\epsilon\}\}$.

Example 2: For $n = 2$, we have

$$\begin{aligned}\mathcal{F}_\mathcal{N} &= \{v_1, v_2, v_3, v_4, v_5, v_6, v_7, \dots\}\\ &= \{\ \epsilon,\ \ 1,\ 2, 11, 12, 21, 22, \dots\}.\end{aligned} \tag{38}$$

Notation 1: The matrix function $\mathcal{H}_k(\mathcal{A}, B, C)$ is defined by

$$\begin{aligned}&\mathcal{H}_k(\mathcal{A}, B, C)\\ &= \begin{bmatrix} CA^{(v_1)}B & CA^{(v_2)}B & \cdots & CA^{(v_l)}B\\ CA^{(v_2)}B & CA^{(v_3)}B & \cdots & CA^{(v_{l+1})}B\\ \vdots & \cdots & \ddots & \vdots\\ CA^{(v_l)}B & CA^{(v_{l+1})}B & \cdots & CA^{(v_{2l-1})}B\end{bmatrix}\\ &= \mathcal{M}_k(\mathcal{A}, C)\mathcal{M}_k^\top(\mathcal{A}^\top, B^\top)\\ &\in \mathbb{R}^{\frac{p(1-n^k)}{1-n}\times\frac{q(1-n^k)}{1-n}}\end{aligned} \tag{39}$$

with

$$\mathcal{M}_k(\mathcal{A}, C) \in \mathbb{R}^{\frac{p\left(1-n^k\right)}{1-n}\times r} \tag{40a}$$

$$\mathcal{M}_k^\top(\mathcal{A}^\top, B^\top) \in \mathbb{R}^{r\times\frac{q\left(1-n^k\right)}{1-n}}. \tag{40b}$$

The existence conditions of a right or left CCIS can be formulated into the following theorem by checking the rank of only one matrix.

Theorem 4: Matrices $A_1, \dots, A_n$ share a nontrivial right CIS constrained by C or $A_1, \dots, A_n$ share a nontrivial left CIS constrained by B if and only if

$$\mathrm{rank}\,(\mathcal{H}_r(\mathcal{A}, B, C)) < r. \tag{41}$$

Proof: We first prove the necessity and then the sufficiency. Assume that $A_1, \dots, A_n$ share a nontrivial right CIS constrained by C. According to Theorem 2, we have

$$\mathrm{rank}\,(\mathcal{M}_r(\mathcal{A}, C)) < r. \tag{42}$$

Recall the Sylvester inequality [33]

$$\mathrm{rank}(X) + \mathrm{rank}(Y) - r \le \mathrm{rank}(XY) \tag{43a}$$

$$\mathrm{rank}(XY) \le \min\{\mathrm{rank}(X),\ \mathrm{rank}(Y)\} \tag{43b}$$

for $X \in \mathbb{R}^{m\times r}$ and $Y \in \mathbb{R}^{r\times n}$.

Based on (39), (42), and the Sylvester inequality of (43b), we derive

$$\begin{aligned}&\mathrm{rank}\,(\mathcal{H}_r(\mathcal{A}, B, C))\\ &\quad= \mathrm{rank}\left(\mathcal{M}_r(\mathcal{A}, C)\,\mathcal{M}_r^\top(\mathcal{A}^\top, B^\top)\right)\\ &\quad\le \min\{\mathrm{rank}(\mathcal{M}_r(\mathcal{A}, C)),\ \mathrm{rank}(\mathcal{M}_r^\top(\mathcal{A}^\top, B^\top))\}\\ &\quad\le \mathrm{rank}\,(\mathcal{M}_r(\mathcal{A}, C)) < r.\end{aligned} \tag{44}$$

To show sufficiency, assume that the inequality of (41) holds. By the Sylvester inequality of (43a), we can obtain

$$\begin{aligned}&\mathrm{rank}\,(\mathcal{M}_r(\mathcal{A}, C)) + \mathrm{rank}\left(\mathcal{M}_r^\top(\mathcal{A}^\top, B^\top)\right) - r\\ &\quad\le \mathrm{rank}\left(\mathcal{M}_r(\mathcal{A}, C)\,\mathcal{M}_r^\top(\mathcal{A}^\top, B^\top)\right).\end{aligned} \tag{45}$$

In view of (39) and (41), we obtain

$$\begin{aligned}&\mathrm{rank}\left(\mathcal{M}_r(\mathcal{A}, C)\,\mathcal{M}_r^\top(\mathcal{A}^\top, B^\top)\right)\\ &\quad= \mathrm{rank}\,(\mathcal{H}_r(\mathcal{A}, B, C)) < r.\end{aligned} \tag{46}$$

With (45) and (46), we have

$$\mathrm{rank}\,(\mathcal{M}_r(\mathcal{A}, C)) + \mathrm{rank}\left(\mathcal{M}_r^\top(\mathcal{A}^\top, B^\top)\right) < 2r. \tag{47}$$

It follows from (40) and (47) that:

$$\mathrm{rank}\,(\mathcal{M}_r(\mathcal{A}, C)) < r \quad \text{or} \tag{48a}$$

$$\mathrm{rank}\left(\mathcal{M}_r^\top(\mathcal{A}^\top, B^\top)\right) < r. \tag{48b}$$

By Theorem 2, we conclude that $A_1, \dots, A_n$ share a nontrivial right CIS constrained by C or $A_1, \dots, A_n$ share a nontrivial left CIS constrained by B. ■

B. Relationship Between CCIS and CE

For a more in-depth understanding of the effectiveness and novelty of the CCIS, we present further comparisons to the representative CE approach.

A vector $\boldsymbol{w}$ is said to be a CE of A_i if and only if $\boldsymbol{w}$ is an eigenvector of every A_i i.e., there exists an eigenvalue λ_i of A_i satisfying $A_i\boldsymbol{w} = \lambda_i\boldsymbol{w}$ for every $i \in \mathcal{N}$ [30], [34].

Theorem 5: If $A_1, \ldots, A_n \in \mathbb{R}^{r\times r}$ have a CE $\boldsymbol{w} \in \mathbb{C}^r$ satisfying

$$C\boldsymbol{w} = \mathbf{0} \tag{49}$$

then $A_1, \ldots, A_n$ admit a CIS constrained by C; however, the reverse may not hold.

Proof: The condition that $A_1, \ldots, A_n$ have a CE $\boldsymbol{w} \in \mathbb{C}^r$ satisfying (49) indicates that there exist eigenvalues $\lambda_i \in \mathbb{C}$ associated with the CE $\boldsymbol{w}$ such that

$$A_i\boldsymbol{w} = \lambda_i\boldsymbol{w},\ i = 1, \ldots, n. \tag{50}$$

Let $\boldsymbol{w}$ and $\lambda_i \in \mathbb{C}$ be expressed as

$$\lambda_i = \alpha_i + j\beta_i \tag{51a}$$

$$\boldsymbol{w} = \boldsymbol{\mu} + j\boldsymbol{\nu} \tag{51b}$$

with $\alpha_i, \beta_i \in \mathbb{R}$, $\boldsymbol{\mu}, \boldsymbol{\nu} \in \mathbb{R}^r$, and j being the imaginary unit. It follows from (50) and (51) that:

$$\begin{aligned} A_i\boldsymbol{\mu} + jA_i\boldsymbol{\nu} &= A_i\boldsymbol{w} = \lambda_i\boldsymbol{w} = (\alpha_i + j\beta_i)(\boldsymbol{\mu} + j\boldsymbol{\nu}) \\ &= (\alpha_i\boldsymbol{\mu} - \beta_i\boldsymbol{\nu}) + j(\beta_i\boldsymbol{\mu} + \alpha_i\boldsymbol{\nu}) \end{aligned} \tag{52}$$

which yields

$$A_i\boldsymbol{\mu} = \alpha_i\boldsymbol{\mu} - \beta_i\boldsymbol{\nu} \tag{53a}$$

$$A_i\boldsymbol{\nu} = \beta_i\boldsymbol{\mu} + \alpha_i\boldsymbol{\nu}. \tag{53b}$$

From (49) and (51b), we obtain

$$C\boldsymbol{\mu} = C\boldsymbol{\nu} = \mathbf{0}. \tag{54}$$

We conclude from (52) and (54) that the space $\mathcal{W} = \mathrm{span}\{\boldsymbol{\mu}, \boldsymbol{\nu}\}$ is a CIS of $A_1, \ldots, A_n$ constrained by C.

However, the reverse is not true. Interestingly, for the matrices A_1, A_2 and C given in (32), A_1 and A_2 have a CIS $\mathcal{W}$ of (34), whereas there is no CE of A_1 and A_2 satisfying (49). ■

Remark 2: The concept of an IS of matrix A is similar, and it is well known that these are subspaces spanned by subsets of the eigenvectors of A [1]. However, as shown by the proof of Lemma 5, completely different from the IS, the CCIS of n matrices $A_1, \ldots, A_n$ is not necessarily spanned by a CE of $A_1, \ldots, A_n$. Therefore, the CCIS can better reveal the structural properties of multiple matrices than the CE.

IV. CCIS and Reducibility

This section applies the established CCIS to multidimensional (n-D) F-M models. Sufficient and necessary conditions and the related algorithm are developed for the reducibility of F-M models utilizing CCIS.

A. CCIS of F-M Model

The F-M model is characterized by [15], [35], [36], [37], [38]

$$\begin{aligned} x(i_1+1, i_2+1, \ldots, i_n+1) &= A_1x(i_1, i_2+1, \ldots, i_n+1) \\ &\quad + \cdots + A_nx(i_1+1, \ldots, i_{n-1}+1, i_n) \\ &\quad + B_1u(i_1, i_2+1, \ldots, i_n+1) + \cdots \\ &\quad + B_nu(i_1+1, \ldots, i_{n-1}+1, i_n), \end{aligned} \tag{55a}$$

$$y(i_1, \ldots, i_n) = Cx(i_1, \ldots, i_n) + Du(i_1, \ldots, i_n) \tag{55b}$$

where $x(i_1, \ldots, i_n) \in \mathbb{R}^r$, $u(i_1, \ldots, i_n) \in \mathbb{R}^q$, and $y(i_1, \ldots, i_n) \in \mathbb{R}^p$ are the (local) state vector, the input vector and the output vector, respectively; r denotes the order; $A_1, \ldots, A_n \in \mathbb{R}^{r\times r}$, $B_1, \ldots, B_n \in \mathbb{R}^{r\times q}$, $C \in \mathbb{R}^{p\times r}$, $D \in \mathbb{R}^{p\times q}$. In the rest of the article, we denote the F-M model of (55) with $\mathcal{A} := \{A_1, \ldots, A_n\}$ and $\mathcal{B} := \{B_1, \ldots, B_n\}$ by $(\mathcal{A}, \mathcal{B}, C, D; r)$.

The transfer function matrix (TFM) of (55) is given by

$$H(z_1, \ldots, z_n) = C\left(I_r - \sum_{i=1}^{n} z_iA_i\right)^{-1}\left(\sum_{i=1}^{n} z_iB_i\right) + D \tag{56}$$

where z_i represents the unit delay (or backward-shift) operator [35], [37], [39].

From the F-M model of (55), we can construct both the left and right CCISs as follows.

Definition 7: For an F-M model $(\mathcal{A}, \mathcal{B}, C, D; r)$, a subspace $\mathcal{W}$ is said to be a right CCIS of $(\mathcal{A}, \mathcal{B}, C, D; r)$ if it is a right CCIS of $A_1, \ldots, A_n$ constrained by C. Dually, a subspace $\mathcal{W}$ is said to be a left CCIS of $(\mathcal{A}, \mathcal{B}, C, D; r)$ if it is a left CCIS of $A_1, \ldots, A_n$ constrained by B with

$$B = \begin{bmatrix} B_1 & \cdots & B_n \end{bmatrix}. \tag{57}$$

Remark 3: Notably, the IS is indeed beneficial to system identification, analysis, and design [2], [3], [6], [7], [8], [9], [10]. However, due to space limitations, the reducibility of systems is investigated based on CCIS in this article. We then have the following result for F-M models.

B. Sufficient Reducibility Conditions for F-M Models

The reducibility to be considered for F-M models is as follows [34].

Definition 8 (Reducibility of F-M models): For a given F-M model $(\mathcal{A}, \mathcal{B}, C, D; r)$, if there is a new $(\hat{\mathcal{A}}, \hat{\mathcal{B}}, \hat{C}, D; \hat{r})$ such that

$$\begin{aligned} &C\left(I_r - \sum_{i=1}^{n} z_kA_k\right)^{-1}\left(\sum_{k=1}^{n} z_kB_k\right) \\ &= \hat{C}\left(I_{\hat{r}} - \sum_{k=1}^{n} z_k\hat{A}_k\right)^{-1}\left(\sum_{k=1}^{n} z_k\hat{B}_k\right),\ \hat{r} < r \end{aligned} \tag{58}$$

then the given F-M model $(\mathcal{A}, \mathcal{B}, C, D; r)$ is reducible.

The inherently complex structural properties of F-M models, such as those involving n different variable directions and corresponding n distinct state matrices $A_1, \ldots, A_n$, make it extremely difficult to derive minimal state-space models. Therefore, generalizing the well-known traditional IS theory to the situation with multiple matrices is a long-standing goal and challenge for many researchers. To this end, motivated by the IS for the traditional case, a natural approach is to directly explore the relationship between the IS and the reducibility of an F-M model. Then, a sufficient condition for the reducibility of an F-M model can be given by utilizing the proposed CCIS, as follows.

Theorem 6: An F-M model $(\mathcal{A}, \mathcal{B}, C, D; r)$ is reducible if it admits a nontrivial right CCIS.

Proof: Assume that $A_1, \ldots, A_n$ share a nontrivial right CCIS $\mathcal{W}$ with a basis $\{\boldsymbol{w}_1, \ldots, \boldsymbol{w}_{\tilde{r}}\}$ with $\tilde{r} \geq 1$. Choose $\hat{r} := r - \tilde{r}$ vectors $\boldsymbol{w}_{\tilde{r}+1}, \ldots, \boldsymbol{w}_r$ to construct a nonsingular matrix

$$\begin{aligned} T &:= \left[\boldsymbol{w}_1 \quad \cdots \quad \boldsymbol{w}_{\tilde{r}} \mid \boldsymbol{w}_{\tilde{r}+1} \quad \cdots \quad \boldsymbol{w}_r\right] \\ &:= \left[T_1 \mid T_2\right] \end{aligned} \tag{59}$$

and define

$$L := \begin{bmatrix} L_1 \\ L_2 \end{bmatrix} := T^{-1} \tag{60}$$

with $L_1 \in \mathbb{R}^{\tilde{r} \times r}$ and $L_2 \in \mathbb{R}^{(r-\tilde{r}) \times r}$.

We derive from the CIS of A_i, $i = 1, \ldots, n$, that $A_i \boldsymbol{w}_k$, $k = 1, \ldots, \tilde{r}$, can be linearly represented by the $\tilde{r}$ vectors $\boldsymbol{w}_1, \ldots, \boldsymbol{w}_{\tilde{r}}$, i.e., there is a matrix $A_{i1} \in \mathbb{R}^{\tilde{r} \times \tilde{r}}$ such that

$$\left[A_i \boldsymbol{w}_1 \quad \cdots \quad A_i \boldsymbol{w}_{\tilde{r}}\right] = \left[\boldsymbol{w}_1 \quad \cdots \quad \boldsymbol{w}_{\tilde{r}}\right] A_{i1}. \tag{61}$$

Since the n vectors $\boldsymbol{w}_1, \ldots \boldsymbol{w}_r \in \mathbb{R}^n$ are independent and are basis vectors for $\mathbb{R}^n$, $A_i \boldsymbol{w}_k \in \mathbb{R}^n$ for all $k \in \{\tilde{r}+1, \ldots, n\}$ must be linearly represented by the $\boldsymbol{w}_1, \ldots \boldsymbol{w}_r \in \mathbb{R}^n$, i.e., there are matrices $A_{i2} \in \mathbb{R}^{\tilde{r} \times (r-\tilde{r})}$ and $A_{i4} \in \mathbb{R}^{(r-\tilde{r}) \times (r-\tilde{r})}$ such that

$$\begin{aligned} &\left[A_i \boldsymbol{w}_{\tilde{r}+1} \quad \cdots \quad A_i \boldsymbol{w}_r\right] \\ &= \left[\boldsymbol{w}_1 \quad \cdots \quad \boldsymbol{w}_{\tilde{r}} \mid \boldsymbol{w}_{\tilde{r}+1} \quad \cdots \quad \boldsymbol{w}_r\right] \begin{bmatrix} A_{i2} \\ A_{i4} \end{bmatrix}. \end{aligned} \tag{62}$$

It follows from (59), (61), and (62) that:

$$\begin{aligned} A_i T &= \left[A_i \boldsymbol{w}_1 \quad \cdots \quad A_i \boldsymbol{w}_{\tilde{r}} \mid A_i \boldsymbol{w}_{\tilde{r}+1} \quad \cdots \quad A_i \boldsymbol{w}_r\right] \\ &= \left[\boldsymbol{w}_1 \quad \cdots \quad \boldsymbol{w}_{\tilde{r}} \mid \boldsymbol{w}_{\tilde{r}+1} \quad \cdots \quad \boldsymbol{w}_r\right] \begin{bmatrix} A_{i1} & A_{i,2} \\ \mathbf{0} & A_{i4} \end{bmatrix} \\ &= T \left[\begin{array}{c|c} A_{i1} & A_{i2} \\ \hline \mathbf{0} & A_{i4} \end{array}\right]. \end{aligned} \tag{63}$$

Premultiplying both sides of the previous equation by $L = T^{-1}$ yields

$$L A_i T = \begin{bmatrix} A_{i1} & A_{i2} \\ \mathbf{0} & A_{i4} \end{bmatrix}. \tag{64}$$

From (59), (60), and (64), we obtain

$$A_{i1} = L_1 A_i T_1, \ A_{i2} = L_1 A_i T_2, \ A_{i4} = L_2 A_i T_2. \tag{65}$$

Because $\mathcal{W}$ is a CIS constrained by C, we derive

$$C T_1 = \left[C \boldsymbol{w}_1 \quad \cdots \quad C \boldsymbol{w}_{\tilde{r}}\right] := \left[\mathbf{0} \quad \cdots \quad \mathbf{0}\right] = \mathbf{0} \tag{66}$$

and then

$$\begin{aligned} CT &= \left[CT_1 \quad CT_2\right] = \left[\mathbf{0} \quad CT_2\right] \\ &:= \left[\mathbf{0} \quad C_2\right]. \end{aligned} \tag{67}$$

Algorithm 2: Reducing an F-M Model Using a Right CCIS.

Input: $(\mathcal{A}, \mathcal{B}, C, D; r)$.
Output: $(\hat{\mathcal{A}}, \hat{\mathcal{B}}, \hat{C}, D; \hat{r})$.
1: $\mathcal{W} = \operatorname{span}\{\boldsymbol{w}_1, \ldots, \boldsymbol{w}_{\tilde{r}}\} \leftarrow$ Algorithm II;
2: Construct a matrix T in the form of (59) and set matrix $L = T^{-1}$;
3: Determine T_2 from the matrix T of (59);
4: Obtain L_2 from the matrix L of (60);
5: Derive a new F-M model $(\hat{\mathcal{A}}, \hat{\mathcal{B}}, \hat{C}, D; \hat{r})$ by (70).

It follows from (60) that:

$$L B_i = \begin{bmatrix} L_1 B_i \\ L_2 B_i \end{bmatrix} := \begin{bmatrix} B_{i1} \\ B_{i2} \end{bmatrix}. \tag{68}$$

On account of (64), (67), and (68), we derive

$$\begin{aligned} &C\left(I_r - \sum_{i=1}^n z_i A_i\right)^{-1} \left(\sum_{i=1}^n z_i B_i\right) \\ &= C T T^{-1} \left(I_r - \sum_{i=1}^n z_i A_i\right)^{-1} L^{-1} L \left(\sum_{i=1}^n z_i B_i\right) \\ &= CT \left(I_r - \sum_{i=1}^n z_i L A_i T\right)^{-1} \left(\sum_{i=1}^n z_i L B_i\right) \\ &= \left[\mathbf{0} \quad C_2\right] \left(I_r - \sum_{i=1}^n z_i \begin{bmatrix} A_{i1} & A_{i2} \\ \mathbf{0} & A_{i4} \end{bmatrix}\right)^{-1} \left(\sum_{i=1}^n z_i \begin{bmatrix} B_{i1} \\ B_{i2} \end{bmatrix}\right) \\ &= C_2 \left(I_{\hat{r}} - \sum_{i=1}^n z_i A_{i4}\right)^{-1} \left(\sum_{i=1}^n z_i B_{i2}\right) \end{aligned} \tag{69}$$

which indicates that a new F-M model $(\hat{\mathcal{A}}, \hat{\mathcal{B}}, \hat{C}, D; \hat{r})$ with lower order $\hat{r} = r - \tilde{r} < r$ is derived with

$$\begin{aligned} &\hat{A}_i = A_{i4} = L_2 A_i T_2, \ \hat{B}_i = B_{i2} = L_2 B_i \\ &\hat{C} = C_2 = C T_2. \end{aligned} \tag{70}$$

■

Remark 4: The existence of a nontrivial right CCIS $\mathcal{W}$ of an F-M model indicates $\tilde{r} = \dim(\mathcal{W}) \geq 1$. By means of the proof of Theorem 6, the order of the new F-M model $(\hat{\mathcal{A}}, \hat{\mathcal{B}}, \hat{C}, D; \hat{r})$ is $\hat{r} = r - \tilde{r}$, which must be less than the order r of the given F-M model.

Based on the proof of Theorem 6, we present a reduction algorithm for the F-M model in Algorithm 2.

Remark 5: Algorithm 2 starts with the right CCIS $\mathcal{W}$ obtained by Algorithm II. Then, the algorithm constructs matrices T and L in Line 2. Lines 3 and 4 extract matrices T_2 and L_2 from T and L, respectively. Finally, a new lower-order F-M model $(\hat{\mathcal{A}}, \hat{\mathcal{B}}, \hat{C}, D; \hat{r})$ is determined by (70) in Line 5. It follows from (70) in the proof of Theorem 6 that the obtained F-M model $(\hat{\mathcal{A}}, \hat{\mathcal{B}}, \hat{C}, D; \hat{r})$ is equivalent to the original one $(\mathcal{A}, \mathcal{B}, C, D; r)$, but with a lower order.

Example 3: To demonstrate the details and effectiveness of Algorithm 2, consider the F-M model $(\mathcal{A}, \mathcal{B}, C, D; r)$ where

$\mathcal{A} = \{A_1, A_2\}$ and C are given in (32), and

$$B_1 = \begin{bmatrix} 2 \\ -2 \\ 3 \\ 1 \end{bmatrix}, \; B_2 = \begin{bmatrix} 0 \\ -2 \\ 4 \\ -1 \end{bmatrix}, \; D = 0. \tag{71}$$

A CIS of A_1 and A_2 constrained by C is given in (34). Then, we can construct a nonsingular matrix as follows:

$$T = \left[\begin{array}{cc|cc} 0 & 1 & 1 & 0 \\ -1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{array}\right] := [T_1 \mid T_2] \tag{72}$$

and set

$$L = T^{-1} = \left[\begin{array}{cccc} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ \hline 1 & 0 & 0 & -1 \\ 0 & 1 & 1 & 0 \end{array}\right] := \begin{bmatrix} L_1 \\ \hline L_2 \end{bmatrix}. \tag{73}$$

According to (60), a new 2-D F-M model $(\hat{\mathcal{A}}, \hat{\mathcal{B}}, \hat{C}, D; \hat{r})$ with lower order is determined by

$$\begin{aligned} \hat{A}_1 &= \begin{bmatrix} 1 & -2 \\ 2 & 1 \end{bmatrix}, \quad \hat{A}_2 = \begin{bmatrix} 1 & -1 \\ -1 & -1 \end{bmatrix} \\ \hat{B}_1 &= \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \quad \hat{B}_2 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}, \quad \hat{C} = \begin{bmatrix} 1 & 1 \end{bmatrix} \\ D &= 0, \quad \hat{r} = 4 - 2 = 2. \end{aligned} \tag{74}$$

The reducibility condition based on the right CCIS of the F-M model has been presented in Theorem 6. In a similar manner, the reducibility condition can be determined based on the left CCIS.

Theorem 7: An F-M model $(\mathcal{A}, \mathcal{B}, C, D; r)$ is reducible if it admits a nontrivial left CCIS.

Proof: Since the proof of Theorem 7 can be proved similarly to that of Theorem 6, we give only the main ideas of the proof.

Assume that $A_1, \ldots, A_n$ share a nontrivial left CCIS $\mathcal{V}$ with a basis $\{\boldsymbol{v}_1, \ldots, \boldsymbol{v}_{\tilde{r}}\}$ with $\tilde{r} \geq 1$. Choose $\hat{r} := r - \tilde{r}$ vectors $\boldsymbol{v}_{\tilde{r}+1}, \ldots, \boldsymbol{v}_r$ to construct a nonsingular matrix

$$\begin{aligned} \hat{T}^\top &:= \left[\boldsymbol{v}_1 \;\; \cdots \;\; \boldsymbol{v}_{\tilde{r}} \mid \boldsymbol{v}_{\tilde{r}+1} \;\; \cdots \;\; \boldsymbol{v}_r\right] \\ &:= \left[\hat{T}_1^\top \mid \hat{T}_2^\top\right] \end{aligned} \tag{75}$$

and define

$$R := \begin{bmatrix} R_1 & R_2 \end{bmatrix} := \hat{T}^{-1} \tag{76}$$

with $\hat{T}_1 \in \mathbb{R}^{\tilde{r} \times r}$, $\hat{T}_2 \in \mathbb{R}^{(r-\tilde{r}) \times r}$, $R_1 \in \mathbb{R}^{r \times \tilde{r}}$, and $R_2 \in \mathbb{R}^{r \times (r-\tilde{r})}$. The given F-M model $(\mathcal{A}, \mathcal{B}, C, D; r)$ can be exactly reduced to the new one $(\hat{\mathcal{A}}, \hat{\mathcal{B}}, \hat{C}, D; \hat{r})$ with

$$\begin{aligned} &\hat{A}_i = \hat{T}_2 A_i R_2, \; \hat{B}_i = \hat{T}_2 B_i, \; \hat{C} = C\hat{T}_2 \\ &\hat{r} = r - \tilde{r} < r. \end{aligned} \tag{77}$$

■

Remark 6: The so-called CE approach in [34] states that a given F-M model $(\mathcal{A}, \mathcal{B}, C, D; r)$ can be reduced if $A_1, \ldots, A_n$ share a CE $\boldsymbol{w}$ such that (49) holds. As clarified in Section III-B, the reducibility condition given in the CE approach [34] always requires the existence of a CCIS; however, the opposite is not necessarily true. As a result, the reducibility conditions in [30] and [34] can only be viewed as the eigenvector-based IS of dimension one. Therefore, the CCIS approach is more general and practical than existing methods.

C. Necessary Reducibility Conditions for F-M Models

This section shows that the sufficient reducibility condition given in the previous section is also necessary in the noncommutative setting (NCS). To this end, the following definitions are required.

Definition 9: A formal power series of the TFM in the NCS is defined by

$$H(z_1, \ldots, z_n) = \sum_{w \in \mathcal{F}_\mathcal{N}} H_w z^{(w)} \tag{78}$$

where H_w is the coefficient matrix w.r.t.

$$z^{(w)} = z_{\alpha_l} z_{\alpha_{l-1}} \cdots z_{\alpha_1} \tag{79}$$

with $w = \alpha_1 \alpha_2 \cdots \alpha_l$. Note that the variables $z_1, \ldots, z_n$ in (78) are not commutative (see, e.g., [40], [41]), i.e.,

$$z_{\alpha_1} z_{\alpha_2} \neq z_{\alpha_2} z_{\alpha_1}$$

for all $\alpha_1 \neq \alpha_2$ and $\alpha_1, \alpha_2 \in \mathcal{N}$.

Now, the TFM of an F-M model in the NCS can be stated as follows.

Definition 10: For the TFM $H(z_1, \ldots, z_n)$ of (56), if one assumes the unit delay operators $z_1, \ldots, z_n$ are noncommutative, i.e., $z_{\alpha_1} z_{\alpha_2} \neq z_{\alpha_2} z_{\alpha_1}$ for $\alpha_1 \neq \alpha_2$ and $\alpha_1, \alpha_2 \in \mathcal{N}$, then such a TFM is called noncommutative. In other words, the noncommutative TFM of the F-M model $(\mathcal{A}, \mathcal{B}, C, D; r)$ is expressed as

$$\begin{aligned} H(z_1, \ldots, z_n) &= C \sum_{k=0}^{\infty} \left(\sum_{i=1}^{n} A_i z_i \right)^k \left(\sum_{i=1}^{n} B_i z_i \right) + D \\ &= \sum_{w \in \mathcal{F}_\mathcal{N}^+} C A^{(v)} B_{\alpha_1} z^{(w)} + D \end{aligned} \tag{80}$$

where $w = \alpha_1 \alpha_2 \cdots \alpha_l = \alpha_1 v \in \mathcal{F}_\mathcal{N}^+$, $A^{(v)}$ and $z^{(w)}$ are defined in (36) and (79), respectively.

Remark 7: One of the primary motivations for exploring the NCS comes from its connection to robust control, as elaborated in [40] and [41]. Specifically, this connection is manifested through the consideration of formal power series in noncommutative indeterminates, particularly in relation to the structured singular value concerning time-variant structured uncertainties modeled by a linear fractional model [40], [41]. Another noteworthy application, emerging more recently, is from a quantum physical interpretation of recursion in the noncommutative F-M model in terms of quantum filtering and quantum tomography [42].

Furthermore, in numerous cases, a commutative result can be attained by shifting to the NCS, leveraging noncommutative theory, and subsequently reverting to the commutative context. A notable illustration of this methodology is found in the seminal work of FM [43], where they employed this approach to establish a state-space realization theorem for rational functions involving multiple commuting variables.

Next, the realization and reduction of the F-M model in the NCS can be stated as follows.

Definition 11: For a given TFM $H(z_1,\ldots,z_n)$ in the form of (78) if there exist matrices $A_1,\ldots,A_n$, $B_1,\ldots,B_n$, C and D such that

$$CA^{(v)}B_{\alpha_1} = H_w,\ D = H(\epsilon) \tag{81}$$

where $w = \alpha_1\alpha_2\cdots\alpha_l = \alpha_1 v \in \mathcal{F}_N^+$, $A^{(v)}$ and z^w are defined in (36) and (79), respectively, then $(\mathcal{A},\mathcal{B},C,D;r)$ is called an F-M model realization of (78) in the NCS.

Definition 12: For a given F-M model $(\mathcal{A},\mathcal{B},C,D;r)$, if there is a new one $(\hat{\mathcal{A}},\hat{\mathcal{B}},\mathcal{C},D;\hat{r})$ satisfying

$$\hat{C}\hat{A}_v\hat{B}_{\alpha_1}z^{(w)} = CA^{(v)}B_{\alpha_1}z^{(w)},\ \hat{r}\le r \tag{82}$$

for all $w = \alpha_1\alpha_2\cdots\alpha_l = \alpha_1 v \in \mathcal{F}_N^+$, $A^{(v)}$ and $z^{(w)}$ being defined in (36) and (79), respectively; then, the given F-M model is reducible.

The necessary reducibility condition for the F-M model in the NCS can be given as follows.

Theorem 8: For an F-M model $(\mathcal{A},\mathcal{B},C,D;r)$, if there is another one $(\hat{\mathcal{A}},\hat{\mathcal{B}},\hat{C},\hat{D};\hat{r})$ with $\hat{r}<r$ such that they share the same noncommutative TFM, then the F-M model $(\mathcal{A},\mathcal{B},C,D;r)$ admits a left or right CCIS.

Proof: Assume that there is a reduced-order F-M model $(\hat{\mathcal{A}},\hat{\mathcal{B}},\hat{C},\hat{D};\hat{r})$ for $(\mathcal{A},\mathcal{B},C,D,r)$ with $\hat{r}<r$, but $(\mathcal{A},\mathcal{B},C,D;r)$ has no left or right CCIS. We derive from the same noncommutative TFMs of $(\mathcal{A},\mathcal{B},C,D;r)$ and $(\hat{\mathcal{A}},\hat{\mathcal{B}},\hat{C},\hat{D};\hat{r})$ that

$$\begin{aligned}\mathcal{H}_r(\mathcal{A},B,C) &= \mathcal{H}_r(\hat{\mathcal{A}},\hat{B},\hat{C})\\ &= \mathcal{M}_r(\hat{\mathcal{A}},\hat{C})\mathcal{M}_r^\top(\hat{A}^\top,B^\top).\end{aligned} \tag{83}$$

Note that $\mathcal{M}_r(\hat{\mathcal{A}},\hat{C}) \in \mathbb{R}^{\frac{p(1-n^{\hat{r}})}{1-n}\times\hat{r}}$; thus, we have

$$\operatorname{rank}\left(\mathcal{M}_r(\hat{\mathcal{A}},\hat{C})\right) \le \hat{r} < r. \tag{84}$$

According to Sylvester inequality (43), we obtain

$$\operatorname{rank}\big(\mathcal{H}_r(\mathcal{A},B,C)\big) = \operatorname{rank}\big(\mathcal{H}_r(\hat{\mathcal{A}},\hat{B},\hat{C})\big) < r. \tag{85}$$

From Theorem 4, we have that the F-M model $(\mathcal{A},\mathcal{B},C,D;r)$ has a left or right CCIS, which contradicts the assumption. ■

V. State-Feedback Control of Rational Parameter Systems

This section proposes an H_∞ gain-scheduled state-feedback controller synthesis for rational parameter systems to further illustrate the superiority of the established CCIS.

A. Discrete-Time Rational Parameter System

Consider the discrete-time rational parameter system (DTRPS) [44]

$$x_{k+1} = A_x(\theta_k)x_k + B_{xu}(\theta_k)u_k + B_{xw}(\theta_k)w_k \tag{86a}$$

$$z_k = C_z(\theta_k)x_k + D_{zu}(\theta_k)u_k + D_{zw}(\theta_k)w_k \tag{86b}$$

where $x_k\in\mathbb{R}^{m_x}$, $u_k\in\mathbb{R}^{m_u}$, $w_k\in\mathbb{R}^{m_w}$, and $z_k\in\mathbb{R}^{m_z}$ are the state vector, control input vector, disturbance vector, and controlled output vector, respectively, and all the matrices are rational and continuous in time-varying parameters collected in $\theta_k\in\Theta=\{(\theta_{k1},\ldots,\theta_{kn})\in\Theta_1\times\cdots\times\Theta_n\}$. Here, Θ_i is the convex hull of the vertices $\theta_i^{[1]},\ldots,\theta_i^{[v_i]}$, i.e., $\theta_{ki}\in\Theta_i$ can be expressed as

$$\theta_{ki} = \sum_{l_i=1}^{v_i}\alpha_{i,l_i}\theta_i^{[l_i]} \tag{87}$$

with

$$\sum_{l_i=1}^{v_i}\alpha_{i,l_i} = 1 \tag{88}$$

and $\alpha_{i,l_i}\ge 0, l_i=1,\ldots,v_i$. We denote by $\mathcal{L}:=\mathcal{L}_1\times\cdots\times\mathcal{L}_n$ the set of all extremal values of the parameters in θ_k, and an element of $\mathcal{L}$ is denoted by $\theta^{[l]}$ with $l=(l_1,\ldots,l_n)$. Then, the total number of vertices in Θ is

$$N_\theta = \prod_{i=1}^{n} v_i.$$

Assuming that the parameter θ_k is available online for controller implementation, we are interested in designing a gain-scheduled state-feedback controller

$$u_k = K(\theta_k)x_k = \left(K_0 + \sum_{i=1}^{n}K_i\theta_{ki}\right)x_k \tag{89}$$

that solves the H_∞-optimal control problem

$$\begin{aligned}&\min_{K(\theta_k),\gamma>0}\ \gamma\\ &\text{s.t.}\quad \|z_k\|\le\gamma\|w_k\|\end{aligned} \tag{90}$$

for the closed-loop system

$$x_{k+1} = A_c(\theta_k)x_k + B_{xw}(\theta_k)w_k \tag{91a}$$

$$z_k = C_c(\theta_k)x_k + D_{zw}(\theta_k)w_k \tag{91b}$$

where the subscript c denotes the closed-loop system and

$$A_c(\theta_k) := A_x(\theta_k) + B_{xu}(\theta_k)K(\theta_k) \tag{92a}$$

$$C_c(\theta_k) := C_z(\theta_k) + D_{zu}(\theta_k)K(\theta_k). \tag{92b}$$

A smaller γ is desirable for better performance [45], [46].

B. Modeling for DTRPSs

The system (86) can be rewritten as

$$\begin{bmatrix}x_{k+1}\\ z_k\end{bmatrix} = G(\theta_k)\begin{bmatrix}x_k\\ u_k\\ w_k\end{bmatrix} \tag{93}$$

with

$$\begin{aligned}G(\theta_k) := &\begin{bmatrix}A_x(\theta_k) & B_{xu}(\theta_k) & B_{xw}(\theta_k)\\ C_z(\theta_k) & D_{zu}(\theta_k) & D_{zw}(\theta_k)\end{bmatrix}\\ &\in\mathbb{R}^{(m_x+m_z)\times(m_x+m_u+m_w)}(\theta_k).\end{aligned} \tag{94}$$

Remark 8: The F-M model realization obtains an F-M model $(\mathcal{A}, \mathcal{B}, C, D; r)$ such that (56) holds from a given multidimensional dimensional (n-D) rational transfer matrix $H(z_1, \ldots, z_n)$, which has already been investigated extensively and intensively (see, e.g., [35], [37]). If we algebraically view the uncertain variable θ_{ki} as the unit delay (or backward-shift) operator z_i, it follows from (56) that the uncertain rational matrix

$$G^\top(\theta_k) = \begin{bmatrix} A_x^\top(\theta_k) & C_z^\top(\theta_k) \\ B_{xu}^\top(\theta_k) & D_{zu}^\top(\theta_k) \\ B_{xw}^\top(\theta_k) & D_{zw}^\top(\theta_k) \end{bmatrix} \in \mathbb{R}^{(m_x+m_u+m_w)\times(m_x+m_z)}(\theta_k) \tag{95}$$

can be expressed as

$$G^\top(\theta_k) = C\left(I_r - \sum_{i=1}^n \theta_{ki} A_i\right)^{-1}\left(\sum_{i=1}^n \theta_{ki} B_i\right) + D. \tag{96}$$

In view of (96), we derive that

$$G^\top(\theta_k) = \begin{bmatrix} D & C \end{bmatrix} \begin{bmatrix} I_{m_x+m_z} & 0 \\ 0 & I_r - \sum_{i=1}^n \theta_{ki} A_i \end{bmatrix}^{-1} \begin{bmatrix} I \\ \sum_{i=1}^n \theta_{ki} B_i \end{bmatrix}. \tag{97}$$

Let

$$\begin{bmatrix} E_{dx} \\ E_{du} \\ E_{dz} \end{bmatrix} := \begin{bmatrix} D & C \end{bmatrix} \tag{98a}$$

$$E_{d\pi}(\theta_k) := \begin{bmatrix} I & 0 \\ 0 & I_r - \sum_{i=1}^n \theta_{ki} A_i \end{bmatrix} \tag{98b}$$

$$\begin{bmatrix} B_{d1}(\theta_k) & B_{d2}(\theta_k) \end{bmatrix} := \begin{bmatrix} I \\ \sum_{i=1}^n \theta_{ki} B_i \end{bmatrix} \tag{98c}$$

with

$$E_{dx} \in \mathbb{R}^{m_x\times(r+m_x+m_z)} \tag{99a}$$
$$E_{du} \in \mathbb{R}^{m_u\times(r+m_x+m_z)} \tag{99b}$$
$$E_{dw} \in \mathbb{R}^{m_w\times(r+m_x+m_z)} \tag{99c}$$
$$E_{d\pi}(\theta_k) \in \mathbb{R}^{(r+m_x+m_z)\times(r+m_x+m_z)}(\theta_k) \tag{99d}$$
$$B_{d1}(\theta_k) \in \mathbb{R}^{(r+m_x+m_z)\times m_x}(\theta_k) \tag{99e}$$
$$B_{d2}(\theta_k) \in \mathbb{R}^{(r+m_x+m_z)\times m_z}(\theta_k). \tag{99f}$$

Here, the subscript d indicates that the transfer function $G^\top(\theta_k)$ is dual of $G(\theta_k)$.

It can be seen from (98) that the parameter-varying matrix functions $E_{d\pi}(\theta_k)$ and $B_{d1}(\theta_k)$ and $B_{d2}(\theta_k)$ are all linear affine on θ_k, i.e., each of these matrices can be represented in the form of

$$\mathcal{M}(\theta_k) = \mathcal{M}_0 + \sum_{i=1}^n \mathcal{M}_i \theta_{ki}. \tag{100}$$

Remark 9: It is interesting to note that using the relation (87), (88) and applying homogenization procedure, which was developed in [47] to an affine matrix $\mathcal{M}(\theta_k)$ of (100), we can obtain

$$\mathcal{M}(\theta_k) = \sum_{l=(1,\ldots,1)}^{(v_1,\ldots,v_n)} \alpha_{1,l_1}\cdots\alpha_{n,l_n}\mathcal{M}^{[l]} \tag{101}$$

where $\mathcal{M}^{[l]}$ are the vertices.

In view of (95), (96), and (98), we derive that

$$A_x^\top(\theta_k) = E_{dx}E_{d\pi}^{-1}(\theta_k)B_{d1}(\theta) \tag{102a}$$
$$B_{xu}^\top(\theta_k) = E_{du}E_{d\pi}^{-1}(\theta_k)B_{d1}(\theta) \tag{102b}$$
$$C_z^\top(\theta_k) = E_{dx}E_{d\pi}^{-1}(\theta_k)B_{d2}(\theta) \tag{102c}$$
$$D_{zu}^\top(\theta_k) = E_{du}E_{d\pi}^{-1}(\theta_k)B_{d2}(\theta) \tag{102d}$$
$$B_{xw}^\top(\theta_k) = E_{dw}E_{d\pi}^{-1}(\theta_k)B_{d1}(\theta) \tag{102e}$$
$$D_{zw}^\top(\theta_k) = E_{dw}E_{d\pi}^{-1}(\theta_k)B_{d2}(\theta). \tag{102f}$$

C. Gain-Scheduled State-Feedback Control of Rational Parameter Systems

The H_∞ performance of (91) is equal to the performance of the dual system defined by

$$x_{d(k+1)} = A_c^\top(\theta_k)x_{dk} + C_c^\top(\theta_k)w_{dk} \tag{103a}$$
$$z_{dk} = B_{xw}^\top(\theta_k)x_{dk} + D_{zw}^\top(\theta_k)w_{dk}. \tag{103b}$$

According to (102a)–(102d), (103a) can be rewritten as

$$\begin{aligned} x_{d(k+1)} &= (E_{dx}(\theta_k) + K^\top(\theta_k)E_{du}(\theta_k)) \\ &\quad \times E_{d\pi}^{-1}(\theta_k)(B_{d1}(\theta_k)x_{dk} + B_{d2}(\theta_k)w_{dk}) \\ &= (E_{dx}(\theta_k) + K^\top(\theta_k)E_{du}(\theta_k))(-\pi_{dk}) \end{aligned} \tag{104}$$

with

$$\pi_{dk} := -E_{d\pi}^{-1}(\theta_k)(B_{d1}(\theta_k)x_{dk} + B_{d2}(\theta_k)w_{dk}). \tag{105}$$

In view of (102e)–(102f), (103b) can be represented as

$$\begin{aligned} z_{d(k+1)} &= \left(E_{dw}E_{d\pi}^{-1}(\theta_k)B_{d1}(\theta)\right)x_{dk} \\ &\quad + \left(E_{dw}E_{d\pi}^{-1}(\theta_k)B_{d2}(\theta)\right)w_{dk} \\ &= -E_{dw}\pi_{dk}. \end{aligned} \tag{106}$$

Let

$$E_d(\theta_k) := \begin{bmatrix} I & 0 & E_{dx} + K^\top(\theta_k)E_{du} & 0 & 0 \\ 0 & I & E_{dw} & 0 & 0 \\ 0 & 0 & E_{d\pi}(\theta_k) & B_{d1}(\theta_k) & B_{d2}(\theta_k) \end{bmatrix} \tag{107a}$$

$$\eta_{dk}^\top := \begin{bmatrix} x_{d(k+1)}^\top & z_{dk}^\top & \pi_{dk}^\top & x_{dk}^\top & w_{dk}^\top \end{bmatrix}. \tag{107b}$$

Then, it follows from (104), (105), and (106) that the system of (103) is equivalent to the following system:

$$E_d(\theta_k)\eta_{dk} = 0. \tag{108}$$

Therefore, the H_∞ performance of the closed-loop system (92) is equal to that of $E_d(\theta_k)\eta_{dk} = 0$.

Define

$$N_{dx}(\theta_k) := \begin{bmatrix} S_{dx}(\theta_k) & 0 & S_{dx}(\theta_k)E_{dx}+S_{dy}(\theta_k)E_{du} & 0 & 0 \end{bmatrix} \tag{109a}$$

$$N_{d\pi}(\theta_k) := \begin{bmatrix} 0 & I & E_{dw} & 0 & 0 \\ 0 & 0 & E_{d\pi}(\theta_k) & B_{d1}(\theta_k) & B_{d2}(\theta_k) \end{bmatrix} \tag{109b}$$

where $S_{dx}(\theta_k)$ and $S_{dy}(\theta_k)$ are affine matrices on θ_k with appropriate sizes. Let $P_d(\theta_k)$ be an affine matrix on θ_k with an appropriate size. $P_d^{[l]}$, $P_d^{[l+]}$, $S_{dx}^{[l]}$, $S_{dy}^{[l]}$, $N_{dx}^{[l]}$, and $N_{d\pi}^{[l]}$ represent the vertices of $P_d(\theta_k)$, $P_d(\theta_{k+1})S_{dx}(\theta_k)$, $S_{dy}(\theta_k)$, $N_{dx}(\theta_k)$, and $N_{d\pi}(\theta_k)$, respectively. Then, the gain-scheduled state-feedback controller for the DTRPS can be given by the following result.

Theorem 9: If there exist N_θ symmetric positive-definite matrices $P_d^{[l]}, P_d^{[l+]}$ and matrix $S_{dx}^{[l]}$, $S_{dy}^{[l]}$, and $S_{d\pi}$ of an appropriate size such that

$$\begin{aligned} &\mathrm{diag}\left(P_d^{[l+]}, \quad I, \quad 0, \quad -P_d^{[l]}, \quad -\gamma^2 I\right) \\ &\prec \left\{\begin{bmatrix} I \\ 0 \end{bmatrix} N_{dx}^{[l]}\right\}^s + \left\{S_{d\pi}N_{d\pi}^{[l]}\right\}^s \end{aligned} \tag{110}$$

hold simultaneously, then the gain-scheduled state-feedback gains

$$K(\theta_k) = \sum_{l=(1,\ldots,1)}^{(v_1,\ldots,v_n)} \alpha_{1,l_1}\cdots\alpha_{n,l_n}K^{[l]} \tag{111}$$

with

$$K^{[l]} = \left(S_{dy}^{[l]}\right)^\top \left(\left(S_{dx}^{[l]}\right)^\top\right)^{-1}$$

guarantee that the closed-loop system of (91) has an H_∞ performance smaller than γ regardless of $\theta \in \Theta$.

Proof: By the convexity of matrix inequalities, we derive that the linear matrix inequalities (LMIs) of (110) hold if and only if

$$\begin{aligned} &\mathrm{diag}\left(P_d(\theta_{k+1}), \quad I, \quad 0, \quad -P_d(\theta_k), \quad -\gamma^2 I\right) \\ &\prec \left\{\begin{bmatrix} I \\ 0 \end{bmatrix} N_{dx}(\theta_k)\right\}^s + \{S_{d\pi}N_{d\pi}(\theta_k)\}^s \end{aligned} \tag{112}$$

for all uncertainties $\theta \in \Theta$. With the change of variable $S_{dy}(\theta_k) = S_{dx}(\theta_k)K^\top(\theta_k)$, we obtain

$$\begin{aligned} &\mathrm{diag}\left(P_d(\theta_{k+1}), \quad I, \quad 0, \quad -P_d(\theta_k), \quad -\gamma^2 I\right) \\ &\prec \{S_d(\theta_k)E_d(\theta_k)\}^s \end{aligned} \tag{113}$$

with

$$S_d(\theta_k) = \left[\begin{array}{c|c} S_{dx}(\theta_k) & S_{d\pi} \\ 0 & \end{array}\right].$$

After congruence operation of $\eta_{dk} \neq 0$ on this last matrix inequality, along the trajectories $(E_d(\theta_k)\eta_{dk} = 0)$

$$\begin{aligned} &x_{d(k+1)}^\top P_d(\theta_{k+1})x_{d(k+1)} - x_{dk}^\top P_d(\theta_k)x_{dk} \\ &+ z_{dk}^\top z_{dk} - \gamma^2\, w_{dk}^\top w_{dk} < 0 \end{aligned} \tag{114}$$

which is sufficient to conclude about asymptotic stability (strictly decreasing Lyapunov function $x_d^\top P_d(\theta_k)x_d$ for zero disturbances) and induced norm performance $\|z_d\|_2 \leq \gamma\|w_d\|_2$ for zero initial conditions. ■

TABLE I
COMPLEXITY OF THE LMI PROBLEMS FOR SEVERAL SIZES OF F-M MODELS

m_x	m_u	m_w	m_z	N_θ	r	N_r	N_v
2	2	2	2	8	5	1088	276
					6	1152	305
					7	1216	336
					8	1280	369
2	2	2	3	8	5	1216	317
					6	1280	349
					7	1344	383
					8	1408	419

Remark 10: It follows from (98), (109), and (110) that the sizes of the matrices $P_d^{[l]}, P_d^{[l+]}, S_{dx}^{[l]}, S_{dy}^{[l]}$ and $S_{d\pi}$ to be solved by the LMIs of (110) are determined by the order of the F-M model. Thus, the complexity of the LMI problems solved by Theorem 9 is directly related to the order of the F-M model. To highlight this point, the total rows of all LMI restrictions N_r and the number of decision variables N_v for the LMIs of Theorem 9 are computed. For an F-M model with order r, we have

$$N_r = N_\theta^2 \times (3m_x + 2m_z + m_w + r) \tag{115a}$$

$$\begin{aligned} N_v &= N_\theta \times \left(1.5\, m_x^2 + m_x m_u + 0.5 m_x\right) + 1 \\ &\quad + (3m_x + 2m_z + m_w + r) \times (m_x + m_z + m_w + r) \end{aligned} \tag{115b}$$

where m_x, $m_u m_w$, and m_z are the sizes of x_k, u_k, w_k, and z_k, respectively, and N_θ is the total number of vertices. To highlight the difference, several values are presented in Table I according to these parameters. It can easily be seen that the higher-order F-M model can lead to large-size problems very quickly. Thus, to reduce the numerical complexity and computation in the LMIs of (110), a lower-order F-M model is obtained by applying the established CCIS approach.

Example 4: To demonstrate the effectiveness of the proposed gain-scheduled state-feedback controller and the CCIS approach, consider the DTRPS of (86) with

$$\begin{aligned} A_x(\theta_k) &= \begin{bmatrix} -\frac{\theta_{k2}}{1+\theta_{k2}} & -\frac{\theta_{k2}}{1+\theta_{k2}} \\ 1 & -\frac{\theta_{k1}+\theta_{k1}\theta_{k2}+\theta_{k3}}{1+\theta_{k1}+\theta_{k2}} \end{bmatrix} \\ B_{xu}(\theta_k) &= \begin{bmatrix} 0 \\ \theta_{k2} \end{bmatrix}, \; B_{xw}(\theta_k) = \begin{bmatrix} \frac{1}{1+\theta_{k2}} \\ 0 \end{bmatrix} \\ C_z(\theta_k) &= \begin{bmatrix} 1 & -\theta_{k1}\theta_{k2} \end{bmatrix} \\ D_{zu}(\theta_k) &= \theta_{k1} + \theta_{k2}, \; D_{zw}(\theta_k) = 1. \end{aligned} \tag{116}$$

The three parameters θ_{k1}, θ_{k2}, and θ_{k3} are in intervals around the nominal value 0 with discrepancies $\delta_1, \delta_2, \delta_3$, i.e., $\theta_{ki} \in [-\delta_i, \delta_i]$, $i = 1, 2, 3$.

To conduct a state feedback design for the DTRPS of (116), we need to obtain a 3-D F-M model for $G^\top(\theta_k)$ in the form of (96). Applying the F-M model realization method of [35] to

$G^\top(\theta_k)$ yields an F-M model with order 10 as shown in (117).

$$A_1 = \begin{bmatrix} 0&0&0&0&0&0&0&0&0&0\\ 0&1&0&0&0&0&0&0&0&0\\ 0&0&-1&-1&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0&0&0\\ 0&0&0&1&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0&1&0 \end{bmatrix},$$

$$A_2 = \begin{bmatrix} -1&0&0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0&0&0\\ 0&0&-1&-1&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0&0&0\\ 0&0&0&1&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0&0&0 \end{bmatrix}$$

$$A_3 = \begin{bmatrix} 0&0&0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0&0&0\\ 0&0&-1&-1&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0&0&0 \end{bmatrix},$$

$$B_1 = \begin{bmatrix} 0&0&0\\ 0&0&0\\ 0&1&0\\ 0&0&0\\ 0&0&0\\ 0&0&0\\ 0&0&0\\ 0&0&1\\ 0&0&0\\ 0&0&0 \end{bmatrix}, B_2 = \begin{bmatrix} 1&0&0\\ 0&0&0\\ 0&0&0\\ 0&1&0\\ 0&0&0\\ 0&0&0\\ 0&0&0\\ 0&0&0\\ 0&0&1\\ 0&0&0 \end{bmatrix},$$

$$B_3 = \begin{bmatrix} 0&0&0\\ 0&0&0\\ 0&0&0\\ 0&0&0\\ 0&1&0\\ 0&0&0\\ 0&0&0\\ 0&0&0\\ 0&0&0\\ 0&0&0 \end{bmatrix}, C = \begin{bmatrix} -1&0&0&0&0&0&0&0&0&0\\ -1&0&1&0&1&1&0&0&0&-1\\ 0&0&0&1&0&1&1&1&1&0\\ -1&0&0&0&0&0&0&0&0&0 \end{bmatrix},$$

TABLE II
COMPARISON OF COMPUTATIONAL COMPLEXITY OF DIFFERENT METHODS

Methods	N_r	N_v	Computing time (s)
Pereira and Oliveira's method in [48]	1472	343	3.1654
Theorem 9 with 10-order F-M model	1216	339	2.3597
Theorem 9 with 6-order F-M model	**960**	**223**	**1.6514**

$$D = \begin{bmatrix} 0&1&1\\ 0&0&0\\ 0&0&0\\ 1&0&1 \end{bmatrix}. \tag{117}$$

To reduce the complexity of the analysis, we can apply the established CCIS approach to obtain a much lower-order F-M model

$$\hat{A}_1 = \begin{bmatrix} 0&0&0&0&0&0\\ 0&-1&-1&0&0&0\\ 0&0&0&0&0&0\\ 0&0&0&0&0&-1\\ 0&0&1&0&0&0\\ 0&0&0&0&0&0 \end{bmatrix}, \hat{B}_1 = \begin{bmatrix} 0&0&0\\ 0&1&0\\ 0&0&0\\ 0&0&-1\\ 0&0&1\\ 0&0&0 \end{bmatrix}$$

$$\hat{A}_2 = \begin{bmatrix} -1&0&0&0&0&0\\ 0&0&0&0&0&0\\ 0&-1&-1&0&0&0\\ 0&0&-1&0&0&0\\ 0&0&1&0&0&0\\ 0&0&0&0&0&0 \end{bmatrix}, \hat{B}_2 = \begin{bmatrix} 1&0&0\\ 0&0&0\\ 0&1&0\\ 0&0&0\\ 0&0&0\\ 0&0&1 \end{bmatrix}$$

$$\hat{A}_3 = \begin{bmatrix} 0&0&0&0&0&0\\ 0&0&0&0&0&0\\ 0&0&0&0&0&0\\ 0&-1&-1&0&0&0\\ 0&0&0&0&0&0\\ 0&0&0&0&0&0 \end{bmatrix}, \hat{B}_3 = \begin{bmatrix} 0&0&0\\ 0&0&0\\ 0&0&0\\ 0&1&0\\ 0&0&0\\ 0&0&0 \end{bmatrix}$$

$$\hat{C} = \begin{bmatrix} -1&0&0&0&0&0\\ -1&1&0&1&1&0\\ 0&0&1&0&1&1\\ -1&0&0&0&0&0 \end{bmatrix}, \hat{D} = \begin{bmatrix} 0&1&1\\ 0&0&0\\ 0&0&0\\ 1&0&1 \end{bmatrix} \tag{118}$$

whose order is 6.

Table II details the total number of rows in all LMI restrictions N_r, the count of scalar decision variables N_v and the computation time. The variables N_r and N_v are crucial for solving the problem and directly impact the overall computational complexity [44]. From Table II, it is evident that the proposed method requires fewer decision variables, fewer total rows, and less computation time compared with the method in [48]. Additionally, the lower-order model exhibits a nearly one-third reduction in both decision variables and total rows compared with the high-order model. This highlights the effective reduction in numerical complexity achieved through lower-order representation.

Table III summarizes the minimum values of H_∞ performance $\gamma_{\min}$ using our design conditions and the method in [48] for different discrepancies $\delta_1 = \delta_2 = \delta_3 = \delta$. Compared with the work in [48], the H_∞ state feedback controller design conditions proposed in this article are significantly less conservative.

TABLE III
COMPARISON OF γ_{MIN} STATE FEEDBACK H_∞ CONTROL

δ	Methods		
	Pereira and Oliveira's method in [48]	Theorem 9 with 10-order F-M model	Theorem 9 with 6-order F-M model
0.211	4.0801	3.4570	**3.4562**
0.212	4.1574	3.4778	**3.4768**
0.213	4.2395	3.4998	**3.4982**
0.214	4.3270	3.5216	**3.5208**
0.215	4.4202	3.5449	**3.5444**
0.216	4.5199	3.5703	**3.5692**
0.217	4.6269	3.5964	**3.5951**
0.218	4.7426	3.6233	**3.6227**
0.219	4.8646	3.6514	**3.6509**
0.220	4.9977	3.6809	**3.6808**
0.221	5.1421	3.7124	**3.7121**
0.222	5.2984	3.7455	**3.7448**
0.223	5.4691	3.7790	**3.7790**
0.224	5.6556	3.8147	**3.8144**
0.225	5.8658	3.8534	**3.8517**
0.226	6.0859	3.8910	**3.8901**
0.227	6.3360	3.9315	**3.9306**
0.228	6.6143	3.9735	**3.9729**
0.229	6.9259	4.0178	**4.0169**
0.230	7.277	4.0634	**4.0631**

Moreover, H_∞ state feedback controllers based on a lower-order representation lead to less conservative than those based on a higher-order representation. All LMIs were formulated by using the YALMIP parser [49] and solved with the SDPT3 solver [50].

Taking $\delta_1 = \delta_2 = \delta_3 = 0.25$ and applying Theorem 9 to the different-order F-M model of (117) and (118) yields state feedback gains, respectively

$$\begin{aligned} K^{(1,1,1)} &= \begin{bmatrix} 3.5143 & -0.7634 \end{bmatrix} \\ K^{(1,1,2)} &= \begin{bmatrix} 3.8194 & -0.1393 \end{bmatrix} \\ K^{(1,2,1)} &= \begin{bmatrix} -1.4230 & 0.2868 \end{bmatrix} \\ K^{(1,2,2)} &= \begin{bmatrix} -1.2821 & 0.1781 \end{bmatrix} \\ K^{(2,1,1)} &= \begin{bmatrix} 1.8696 & 0.1414 \end{bmatrix} \\ K^{(2,1,2)} &= \begin{bmatrix} 2.1379 & 0.8331 \end{bmatrix} \\ K^{(2,2,1)} &= \begin{bmatrix} -2.7885 & 0.1516 \end{bmatrix} \\ K^{(2,2,2)} &= \begin{bmatrix} -2.4772 & 0.0128 \end{bmatrix} \end{aligned} \tag{119}$$

and

$$\begin{aligned} \hat{K}^{(1,1,1)} &= \begin{bmatrix} 3.5111 & -0.7332 \end{bmatrix} \\ \hat{K}^{(1,1,2)} &= \begin{bmatrix} 3.7983 & -0.0719 \end{bmatrix} \\ \hat{K}^{(1,2,1)} &= \begin{bmatrix} -1.2684 & 0.3513 \end{bmatrix} \\ \hat{K}^{(1,2,2)} &= \begin{bmatrix} -0.9815 & 0.1507 \end{bmatrix} \\ \hat{K}^{(2,1,1)} &= \begin{bmatrix} 1.8231 & 0.2305 \end{bmatrix} \\ \hat{K}^{(2,1,2)} &= \begin{bmatrix} 2.0948 & 0.8694 \end{bmatrix} \\ \hat{K}^{(2,2,1)} &= \begin{bmatrix} -2.6228 & 0.2286 \end{bmatrix} \\ \hat{K}^{(2,2,2)} &= \begin{bmatrix} -2.3534 & 0.0960 \end{bmatrix}. \end{aligned} \tag{120}$$

VI. CONCLUSION

The notion of CCIS has been proposed to take into account the overall structural characteristics of all state matrices. The necessary and sufficient conditions for the existence of CCIS have been established, and the corresponding calculation algorithm has been given. It has been proved that the existence of a CE leads to the existence of CCIS, but not vice versa, so the established CCIS can better reveal the internal structural properties of multiple matrices than the CE. Based on this CCIS, the necessary and sufficient reducibility conditions and the associated reduction algorithms for the F-M model have been developed. Moreover, a gain-scheduled state-feedback control has been proposed for rational parameter systems, the numerical complexity of which can be greatly reduced by applying CCIS. Examples have been provided to highlight the details and effectiveness of the new approach.

ACKNOWLEDGMENT

The authors would like to thank the anonymous reviewers for their constructive comments, which helped to improve the article significantly.

REFERENCES

[1] B. Moore and A. Laub, "Computation of supremal (A, B)-invariant and controllability subspaces," *IEEE Trans. Autom. Control*, vol. TAC-23, no. 5, pp. 783–792, Oct. 1978.
[2] C. Huang, G. Feng, H. Zhang, and Z. Wang, "System identification based on invariant subspace," *IEEE Trans. Autom. Control*, vol. 67, no. 3, pp. 1327–1341, Mar. 2022.
[3] C. Huang, "A combined invariant-subspace and subspace identification method for continuous-time state-space models using slowly sampled multi-sine-wave data," *Automatica*, vol. 140, 2022, Art. no. 110261.
[4] C. Huang, "Identification of continuous-time systems with irregular samplings by invariant-subspace based method," *IEEE Trans. Autom. Control*, vol. 69, no. 4, pp. 2406–2413, Apr. 2024, doi: 10.1109/TAC.2023.3304612.
[5] K. Gu and H. Phan-Van, "External direct sum invariant subspace and decomposition of coupled differential-difference equations," *IEEE Trans. Autom. Control*, vol. 69, no. 2, pp. 1022–1028, Feb. 2024, doi: 10.1109/TAC.2023.3304612.
[6] G. Basile and G. Marro, *Controlled and Conditioned Invariants in Linear System Theory*. Englewood Cliffs, NJ, USA: Prentice-Hall, 1992.
[7] G. Zhang, P. Zheng, S. Yu, H. Trinh, and P. Shi, "Controllability analysis and verification for high-order DC-DC converters using switched linear systems theory," *IEEE Trans. Power Electron.*, vol. 36, no. 8, pp. 9678–9688, Aug. 2021.
[8] F. Lin, L. Y. Wang, W. Chen, and M. P. Polis, "On observability of hybrid systems," *IEEE Trans. Autom. Control*, vol. 67, no. 11, pp. 6074–6081, Nov. 2022.
[9] X. Chen and M. Huang, "Linear-quadratic mean field control: The invariant subspace method," *Automatica*, vol. 107, pp. 582–586, 2019.
[10] G. Balas, J. Bokor, and Z. Szabó, "Invariant subspaces for LPV systems and their applications," *IEEE Trans. Autom. Control*, vol. 48, no. 11, pp. 2065–2069, Nov. 2003.
[11] P. J. Antsaklis and A. N. Michel, *A Linear Systems Primer*. Boston, MA, USA: Birkhäuser, 2007.
[12] H. L. Trentelman, A. A. Stoorvogel, and M. Hautus, *Control Theory for Linear Systems*. London, U.K: Springer, 2012.
[13] L. Yang, Z. Fei, Z. Wang, and C. K. Ahn, "Zonotope-based interval estimation for 2-D FMLSS systems using an event-triggered mechanism," *IEEE Trans. Autom. Control*, vol. 68, no. 3, pp. 1655–1666, Mar. 2023.
[14] S. Yan, D. Zhao, H. Wang, S. Matsushita, and L. Xu, "A novel constructive procedure to low-order Fornasini–Marchesini model realization," *J. Franklin Inst.*, vol. 357, no. 3, pp. 1764–1789, 2020.
[15] R. Yang and W. X. Zheng, "Two-dimensional sliding mode control of discrete-time Fornasini–Marchesini systems," *IEEE Trans. Autom. Control*, vol. 64, no. 9, pp. 3943–3948, Sep. 2019.

[16] E. Fornasini, "A. 2-D systems approach to river pollution modelling," *Multidimensional Syst. Signal Process.*, vol. 2, no. 3, pp. 233–265, 1991.

[17] M. G. B. Sumanasena and P. H. Bauer, "Realization using the Fornasini-Marchesini model for implementations in distributed grid sensor networks," *IEEE Trans. Circuits Syst. I, Reg. Papers*, vol. 58, no. 11, pp. 2708–2717, Nov. 2011.

[18] A. N. Vargas, C. M. Agulhari, R. C. L. F. Oliveira, and V. M. Preciado, "Robust stability analysis of linear parameter-varying systems with Markov jumps," *IEEE Trans. Autom. Control*, vol. 67, no. 11, pp. 6234–6239, Nov. 2022.

[19] T. Besselmann, J. Lofberg, and M. Morari, "Explicit MPC for LPV systems: Stability and optimality," *IEEE Trans. Autom. Control*, vol. 57, no. 9, pp. 2322–2332, Sep. 2012.

[20] J. Arroyo, A. Athreya, J. Cape, G. Chen, C. E. Priebe, and J. T. Vogelstein, "Inference for multiple heterogeneous networks with a common invariant subspace," *J. Mach. Learn. Res.*, vol. 22, no. 1, pp. 1–49, Jan. 2021.

[21] S. Gao and P. E. Caines, "Subspace decomposition for graph on LQR: Applications to VLSNs of harmonic oscillators," *IEEE Trans. Control Netw. Syst.*, vol. 8, no. 2, pp. 576–586, Jun. 2021.

[22] G. Pastuszak, "The common invariant subspace problem and Tarski's theorem," *Electron. J. Linear Algebra*, vol. 32, pp. 343–356, 2017.

[23] D. Arapura and C. Peterson, "The common invariant subspace problem: An approach via Gröbner bases," *Linear Algebra Appl.*, vol. 384, pp. 1–7, 2004.

[24] D. Shemesh, "Common eigenvectors of two matrices," *Linear Algebra Appl.*, vol. 62, pp. 11–18, 1984.

[25] G. Pastuszak, "A constructive proof of Tarski's theorem on quantifier elimination in the theory of ACF," 2016, doi: 10.48550/arXiv.1607.05505.

[26] A. Jamiołkowski and G. Pastuszak, "Generalized Shemesh criterion, common invariant subspaces and irreducible completely positive superoperators," *Linear Multilinear Algebra*, vol. 63, no. 2, pp. 314–325, 2015.

[27] Y. A. Al'pin and K. D. Ikramov, "Rational procedures in the problem of common invariant subspaces of two matrices," *J. Math. Sci.*, vol. 114, no. 6, pp. 1757–1764, 2003.

[28] Z. Lin, L. Xu, and N. K. Bose, "A tutorial on Gröbner bases with applications in signals and systems," *IEEE Trans. Circuits Syst. I, Reg. Papers*, vol. 55, no. 1, pp. 445–461, Feb. 2008.

[29] T. Sasaki, "A bridge between Euclid and Buchberger: (An attempt to enhance Gröbner basis algorithm by PRSs and GCDs)," *ACM Commun. Comput. Algebra*, vol. 56, no. 3, pp. 97–124, 2023.

[30] D. Zhao, S. Yan, S. Matsushita, and L. Xu, "Common eigenvector approach to exact order reduction for Roesser state-space models of multidimensional systems," *Syst. Control Lett.*, vol. 134, 2019, Art. no. 104559.

[31] M. Petreczky, L. Bako, and J. H. V. Schuppen, "Realization theory of discrete-time linear switched systems," *Automatica*, vol. 49, no. 11, pp. 3337–3344, 2013.

[32] M. Kreuzer, A. Miasnikov, and F. Walsh, "Decomposing finite $\mathbb{Z}$-algebras," 2023, doi: 10.48550/arXiv.2308.01735.

[33] A. R. Horn and C. R. Johnson, *Matrix Analysis*. Cambridge, U.K.: Cambridge Univ. Press, 2012.

[34] D. Zhao, S. Yan, S. Matsushita, and L. Xu, "Common eigenvector approach to exact order reduction for multidimensional Fornasini–Marchesini state-space models," *Int. J. Syst. Sci.*, vol. 50, no. 1, pp. 60–74, 2019.

[35] D. Zhao, S. Yan, S. Matsushita, and L. Xu, "An approach to multidimensional Fornasini-Marchesini state-space model realization w.r.t. columns of transfer matrices," *Syst. Control Lett.*, vol. 123, pp. 116–123, 2019.

[36] D. Zhao, C. K. Ahn, W. Paszke, F. Fu, and Y. Li, "Fault diagnosability analysis of two-dimensional linear discrete systems," *IEEE Trans. Autom. Control*, vol. 66, no. 2, pp. 826–832, Feb. 2021.

[37] S. Yan, D. Zhao, H. Wang, and L. Xu, "Elementary operation approach to Fornasini-Marchesini state-space model realization of multidimensional systems," *J. Franklin Inst.*, vol. 357, no. 12, pp. 8098–8117, 2020.

[38] W. Ren and J. Xiong, "Robust filtering for 2-D discrete-time switched systems," *IEEE Trans. Autom. Control*, vol. 66, no. 10, pp. 4747–4760, Oct. 2021.

[39] S. Yan, L. Xu, Q. Zhao, and Y. Tian, "Elementary operation approach to order reduction for Roesser state-space model of multidimensional systems," *IEEE Trans. Circuits Syst. I: Reg. Papers*, vol. 61, no. 3, pp. 789–802, Mar. 2014.

[40] C. L. Beck, J. Doyle, and K. Glover, "Model reduction of multidimensional and uncertain systems," *IEEE Trans. Autom. Control*, vol. 41, no. 10, pp. 1466–1477, Oct. 1996.

[41] C. L. Beck and J. Doyle, "A necessary and sufficient minimality condition for uncertain systems," *IEEE Trans. Autom. Control*, vol. 44, no. 10, pp. 1802–1813, Oct. 1999.

[42] R. Gohm, "Weak Markov processes as linear systems," *Math. Control Signals Syst.*, vol. 27, no. 3, pp. 375–413, 2015.

[43] E. Fornasini and G. Marchesini, "Doubly-indexed dynamical systems: State-space models and structural properties," *Math. Syst. Theory*, vol. 12, no. 1, pp. 59–72, 1978.

[44] D. Peaucelle, Y. Ebihara, and Y. Hosoe, "Robust observed-state feedback design for discrete-time systems rational in the uncertainties," *Automatica*, vol. 76, pp. 96–102, 2017.

[45] H. Kokame, H. Kobayashi, and T. Mori, "Robust H_∞ performance for linear delay-differential systems with time-varying uncertainties," *IEEE Trans. Autom. Control*, vol. 43, no. 2, pp. 223–226, Feb. 1998.

[46] Y. Wan, D. E. Shen, S. Lucia, R. Findeisen, and R. D. Braatz, "A polynomial chaos approach to robust static output-feedback control with bounded truncation error," *IEEE Trans. Autom. Control*, vol. 68, no. 1, pp. 470–477, Jan. 2023.

[47] R. C. L. F. Oliveira, P.-A. Bliman, and P. L. D. Peres, "Robust LMIs with parameters in multi-simplex: Existence of solutions and applications," in *Proc. IEEE Conf. Decis. Control*, 2008, pp. 2226–2231.

[48] R. L. Pereira and M. S. de Oliveira, "Improved $\mathcal{H}_2$ and $\mathcal{L}_2$ control designs for discrete-time LPV/LFR systems," *Int. J. Control*, vol. 96, no. 7, pp. 1786–1796, 2023.

[49] J. Lofberg, "YALMIP: A toolbox for modeling and optimization in MATLAB," in *Proc. IEEE Int. Conf. Robot. Automat.*, 2004, pp. 284–289.

[50] K.-C. Toh, M. J. Todd, and R. H. Tütüncü, "SDPT3 - A MATLAB software package for semidefinite programming, version 1.3," *Optim. Methods Softw.*, vol. 11, no. 1/4, pp. 545–581, 1999.

Dongdong Zhao (Member, IEEE) received the B.Eng. degree in engineering from Zhengzhou University, Zhengzhou, China, in 2012, the M.Eng. degree in engineering from Lanzhou University, Lanzhou, China, 2016, and the Dr. Eng. degree in systems science from Akita Prefectural University, Akita, Japan, in 2019.

He is currently an Associate Professor with the School of Information Science and Engineering, Lanzhou University. His research interests include uncertain dynamical systems, robotics, machine learning, and multidimensional system theory.

Yu Kang (Senior Member, IEEE) received the Dr. Eng. degree in control theory and control engineering from the University of Science and Technology of China, Hefei, China, in 2005.

From 2005 to 2007, he was a Postdoctoral Fellow with the Academy of Mathematics and Systems Science, Chinese Academy of Sciences, Beijing, China. He is currently a Professor with the Department of Automation, University of Science and Technology of China. His current research interests include adaptive/robust control, variable structure control, mobile manipulators, and Markovian jump systems.

Yun-Bo Zhao (Senior Member, IEEE) received the B.Sc. degree in mathematics from Shandong University, Jinan, China, in 2003, the M.Sc. degree in systems sciences from the Key Laboratory of Systems and Control, Chinese Academy of Sciences, Beijing, China, in 2007, and the Ph.D. degree in control engineering from the University of South Wales, Pontypridd, U.K., in 2008.

He is currently a Professor with the Department of Automation, University of Science and Technology of China, and also with the Institute of Artificial Intelligence, Hefei Comprehensive National Science Center, Hefei, China. He has authored or coauthored more than 100 research articles as well as 2 English monographs on networked intelligent control and 1 Chinese monograph on human–machine autonomy. His current research interests include AI-driven control and automation, specifically, AI-driven networked intelligent control, AI-driven human–machine autonomy, and AI-driven machine gaming.

Li Xu (Senior Member, IEEE) received the B.Eng. degree in engineering from the Huazhong University of Science and Technology, Wuhan, China, in 1982, and the M.Eng. and Dr. Eng. degrees in engineering from the Toyohashi University of Technology, Toyohashi, Japan, in 1990 and 1993, respectively.

From 1993 to 1998, he was an Assistant Professor with the Department of Knowledge-Based Information Engineering, Toyohashi University of Technology. From 1998 to 2000, he was a Lecturer with the Department of Information Management, Asahi University, Gifu, Japan. He is currently a Professor with the Department of Intelligent Mechatronics, Faculty of Systems Science and Technology, Akita Prefectural University, Akita, Japan. His research interests include multidimensional system theory, signal processing, and the application of computer algebra to systems theory.

Dr. Xu was an Associate Editor from 2000 to 2014 and has been an Editorial Board Member since 2014 for the international journal of *Multidimensional Systems and Signal Processing*.

Shi Yan (Member, IEEE) received the B.S. and M.Eng. degrees in engineering from Lanzhou University, Lanzhou, China, in 2001 and 2004, respectively, and the Dr. Eng. degree in systems science from Akita Prefectural University, Akita, Japan, in 2010.

From 2004 to 2007, he was an Assistant Professor with the School of Information Science and Engineering, Lanzhou University. He was a Visiting Researcher in 2007 and a Visiting Research Fellow in 2010 with the Department of Electronics and Information Systems, Akita Prefectural University. He was a Lecturer from 2007 to 2012, an Associate Professor from 2012 to 2020, and is currently a Professor with the School of Information Science and Engineering, Lanzhou University. His research interests include uncertain dynamical systems, robotics, machine learning, and multidimensional systems theory.